Regularities of Fermion Masses and Mixing Angles and Their Extension to the Fourth Generation

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(Received 10 September 1985)

The success of the Fritzsch mass-matrix *Ansatz* in reproducing the observed pattern of the Kobayashi-Maskawa mixings can be explained by having the mass matrices of the charge $\frac{2}{3}$ and $-\frac{1}{3}$ quarks closely proportional to each other. Thus except for light quarks, the ratio of these two masses in each generation should be the same. Extending this to the fourth generation and using the electroweak ρ-parameter constraint we expect that the fourth charged lepton and the seventh (down type) and the eighth (up type) quarks, if they exist, are likely to have masses around 25, 60, and 450 GeV, respectively.

PACS numbers: 12.10.Ck, 14.60.Jj, 14.80.Dq

Phenomenologically, the standard SU(3) $\otimes$ SU(2) $\otimes$ U(1) gauge theory with three generations of fermions has been very successful. However, an understanding of fermion family replication, masses, and mixing angles still eludes us. Presumably, they will serve as important clues in our search for the more basic theory from which the standard model can be derived as the low-energy effective theory. Thus, any simple model that can account for the observed systematics of quark and lepton masses and mixing angles should be very useful. It will also provide basis for any speculation on the existence of a sequential fourth-generation fermions.[1]

Recent results of b-quark lifetime measurements and $(b \to u)/(b \to c)$ branching-ratio limits can be translated into values of the Kobayashi-Maskawa[2] (KM) matrix elements as[3]

$$U = \begin{pmatrix} 0.973 \pm 0.0024 & 0.225 \pm 0.005 & < 0.009 \\ 0.24 \pm 0.03 & 0.82 \pm 0.13 & 0.058 \pm 0.009 \\ \cdots & \cdots & \cdots \end{pmatrix}, \tag{1}$$

where the columns are d, s, and b and the rows are u, c, and t. Furthermore, certain aspects of the $\bar{p}p$ collision data can be interpreted as possible production of the t quark with a mass in the range of 30 to 50 GeV.[4]

As first suggested by Wolfenstein,[5] a particularly useful way to organize the KM matrix elements with their disparate magnitudes is to express them as powers of the Cabibbo angle $\lambda = U_{us} \simeq 0.025$. After imposing the requirements of unitarity, the KM matrix of (1) can be written, up to an $O(\lambda^4)$ correction, as

$$U = \begin{pmatrix} 1-\lambda^2/2 & \lambda & (\sigma - i\eta)A\lambda^3 \\ -\lambda & 1-\lambda^2/2 & A\lambda^2 \\ (1-\sigma-i\eta)A\lambda^3 & -A\lambda^2 & 1 \end{pmatrix} \tag{2}$$

with $A \simeq 1.15$ and $(\sigma^2+\eta^2)^{1/2} < 0.7$. As the quark-mass values[6] in different generations display a hierarchical structure,

$$m_d \simeq 8.9 \text{ MeV}, \quad m_s \simeq 175 \text{ MeV}, \quad m_b \simeq 5.3 \text{ GeV}, \tag{3a}$$
$$m_u \simeq 5.1 \text{ MeV}, \quad m_c \simeq 1.35 \text{ GeV}, \quad m_t \simeq 30\text{–}50 \text{ GeV},$$

we will parametrize them also in terms of λ:

$$\begin{aligned} &m_t : m_c : m_u = 1 : c_t\lambda^2 : u_t\lambda^6, \\ &m_b : m_s : m_d = 1 : s_b\lambda^2 : d_b\lambda^4, \end{aligned} \tag{3b}$$

where $c_t = (m_c/m_t)/\lambda^2$ and similarly defined u_t, s_b, and d_b are $O(1)$ coefficients. Thus, the mass of each succeeding generation increases by λ^{-2}. This pattern is broken only by the "anomalously light" u quark.

We shall study the implications of these hierarchies of masses and mixings on the quark mass matrices, defined with respect to weak eigenstates. The up-type and the down-type mass matrices M^u and M^d are diagonalized by some biunitary transformations:

$$V_L^u M^u V_R^{u\dagger} = \hat{M}^u, \quad V_L^d M^d V_R^{d\dagger} = \hat{M}^d,$$

where $\hat{M}^u$ and $\hat{M}^d$ are diagonal and real. The KM matrix U is simply the product $U = V_L^u V_L^{d\dagger}$. Note that in the limit of $V_L^u = V_L^d$ we have $U = 1$.

What pattern of regularity for M^u and M^d do the observed values of quark masses and KM mixing angles imply? One way to interpret the fact that the KM matrix is close to the unit matrix, $U = 1 + O(\lambda)$, is that the up and down mass matrices are almost proportional to each other.[7] Namely, the matrices normalized to the largest eigenvalues are nearly equal:

$$M^d/m_b = M^u/m_t + \Delta,$$

where the correction term Δ is expected to be $O(\lambda)$. And it can be shown[8,9] that for the case of Hermitian mass matrices the observed hierarchical structure of U_{ij} in Eq. (2) implies that Δ is of even higher order, $O(\lambda^2)$.

The close proportionality of the up and down matrices has another important physics ramification. In the limit $\Delta=0$, we would also have the proportionality of their eigenvalues,

$$m_u/m_d = m_c/m_s, \tag{4a}$$

$$m_t/m_b = m_c/m_s. \tag{4b}$$

Namely, the charge $\frac{2}{3}$ and $-\frac{1}{3}$ quark-mass ratio in each generation should be the same:

$$m_{(2/3)}/m_{(-1/3)} = \text{generation independent}. \tag{5}$$

For quark masses given in (3a) we see that Eq. (4b) is a better approximation than Eq. (4a). This just reflects the fact that the first-generation masses are small and the perturbation Δ will give larger corrections for (4a) than for (4b).[10] We will take the correction to be of the form

$$(m_d/m_s)^{1/2} - (m_u/m_c)^{1/2} = O(\lambda), \tag{6a}$$

$$(m_s/m_b)^{1/2} - (m_c/m_t)^{1/2} = O(\lambda^2). \tag{6b}$$

We shall next show that (6a) and (6b) are precisely the key ingredient that is needed for the Fritzsch *Ansatz* to yield a set of KM angles in close agreement with the observed pattern.

The Fritzsch *Ansatz*[11] for the mass matrix states that only the heaviest generation has a diagonal element and all other lighter masses arise through mixings between neighboring families. We have, for $a=u,\ d$,

$$M^a = P^a F^a Q^a \text{ with } F^a = \begin{pmatrix} 0 & A^a & 0 \\ A^a & 0 & B^a \\ 0 & B^a & C^a \end{pmatrix},$$

where P^a and Q^a are diagonal phase matrices. The real symmetric matrix F^a can be diagonalized by orthogonal transformations $O^a\, F^a\, \tilde{O}^a = \hat{M}^a$. The KM matrix is thus the product

$$U = XO^d P \tilde{O}^u Y, \tag{7}$$

where the diagonal phase matrix $P=P^u P^{d\dagger}$, and the other two matrices X and Y represent the rephasing freedom of the KM elements through the redefinition of the quark phases. Since a Fritzsch mass matrix has only three nonzero elements they can be expressed in terms of the three eigenvalues, and the KM angles U_{ij} can then be computed from quark masses m_i and the phases. It has been shown[12] that the exact formulas for U_{ij} thus obtained can reproduce the phenomenological pattern of (1). In this work we will use the quark-mass hierarchy (3b) to obtain an approximate form for U_{ij}, and show that it is the close interplay between the Fritzsch *Ansatz* and the new relation (5) that generates the correct mixing hierarchy as displayed in (2).

For $m_3 \gg m_2 \gg m_1$, we have the approximate result that $A \simeq (m_1 m_2)^{1/2}$, $B \simeq (m_2 m_3)^{1/2}$, $C \simeq m_3$, and

$$O_{ij} = \begin{pmatrix} 1-(m_1/2m_2) & (m_1/m_2)^{1/2} & -(m_1/m_3)^{1/2} \\ -(m_1/m_2)^{1/2} & 1-(m_1/2m_2)-(m_2/2m_3) & -(m_2/m_3)^{1/2} \\ (m_1/m_3)^{1/2}(m_2/m_3) & (m_2/m_3)^{1/2} & 1-(m_2/2m_3) \end{pmatrix}. \tag{8}$$

This is to be applied for both O^u and O^d, and the KM mixings can then be obtained from Eq. (7). Let us first consider the simple case where all phases are set equal to zero. Using Eqs. (8) and (7) we obtain

$$U_{us} \simeq (m_u/m_c)^{1/2} - (m_d/m_s)^{1/2},$$

$$U_{cb} \simeq -(m_c/m_t)^{1/2} + (m_s/m_b)^{1/2}.$$

Through Eqs. (6a) and (6b) we see that they have just the right magnitudes as in (2). We will now demonstrate that a simple choice of phases can be made to preserve this desired cancellation together with the attractive possibility of "maximal CP nonconservation."

If we start with the definition of phase matrices $X=\mathrm{diag}(e^{i(\alpha_k-\beta_k)})$, $Y=\mathrm{diag}(e^{-i\alpha_k})$, $P=\mathrm{diag}(e^{i\gamma_k})$, Eq. (7) has the components

$$U_{ij} = \left(\textstyle\sum_k O^u_{ik} O^d_{jk} e^{i\gamma_k}\right) e^{-i\beta_i} e^{i(\alpha_i-\alpha_j)}.$$

We make the following phase choices and assumptions: (i) $\beta_i=\gamma_i$, $\alpha_2=\alpha_3$, and $\alpha_1=\alpha_2+\pi$ so that, except for the possible mass-matrix phases γ_i, all the U_{ij} elements will be real (and certain elements such as U_{us} change sign). (ii) $\gamma_2=\gamma_3$ to preserve the cancellation in U_{cb} as discussed above; and $\gamma_1=\gamma_2-\pi/2$ to obtain a large *CP*-nonconserving phase (and to get better agreement of the Cabibbo angle U_{us} in terms of quark masses). The Wolfenstein parameters λ, A, σ, and η can all be computed in terms of quark masses:

$$\begin{aligned} U_{us} &= (m_d/m_s)^{1/2} - i(m_u/m_c)^{1/2} \simeq (m_d/m_s)^{1/2} e^{i\phi}, \\ U_{cb} &= (m_s/m_b)^{1/2} - (m_c/m_t)^{1/2} = A\lambda^2, \\ U_{ub} &= -(m_d/m_b)^{1/2}(m_s/m_b) - iA\lambda^2 (m_u/m_c)^{1/2} \\ &= A\lambda^3(\sigma - i\eta). \end{aligned} \tag{9}$$

Since numerically we have $\lambda=(m_d/m_s)^{1/2}$, the U_{us} phase is small, $\phi=(m_u m_s/m_c m_d)^{1/2}=O(\lambda)$. Thus a rephasing operation to make U_{us} real as in the Wolfenstein parametrization will not affect the leading expression for other elements. We have the prediction that[13]

$$\sigma \simeq -(m_s/m_d)^{3/2}/|U_{cb}| \simeq -0.10,$$

$$\eta \simeq (m_u m_s/m_c m_d)^{1/2} \simeq 0.26.$$

Thus $|U_{ub}/U_{cb}| = \lambda(\sigma^2+\eta^2)^{1/2} \simeq 0.06$ and for the quark-mass values quoted in Eq. (3a) this yields

$$\frac{\Gamma(b\to ue\bar{\nu})}{\Gamma(b\to ce\bar{\nu})} = \frac{1}{0.6}\left|\frac{U_{ub}}{U_{cb}}\right|^2 = 0.006,$$

which is to be compared to the present upper limit of 0.04.[14]

Let us now consider the implications of the statement (5) for the fourth-generation quark masses. If a fourth generation of sequential fermions[15] exists,

$$\begin{pmatrix} o \\ h \end{pmatrix}_L, \quad \begin{pmatrix} \nu_\kappa \\ \kappa \end{pmatrix}_L,$$

then the contribution to the W and Z masses arising from loops of these fermions will give the following correction[16] to the relationship $\rho = M_w^2/M_z^2\cos^2\theta_w = 1$:

$$\Delta\rho = (G_F/8\sqrt{2}\pi^2)[3F(m_o,m_h) + F(m_\nu,m_\kappa)],$$

$$F(m_1,m_2) = m_1^2 + m_2^2 - \frac{2m_1^2m_2^2}{m_1^2-m_2^2}\ln\frac{m_1^2}{m_2^2}.$$

We shall assume that the lepton contribution is negligible [$m_\nu = 0$, $m_h/m_\kappa \simeq 2$ to 3, thus $F(m_o,m_h) >> F(0,m_\kappa)$]. Because m_o/m_h is fixed by Eq. (5), the experimental bound on $\rho - 1$ can be converted into bounds on quark masses. There are a number of uncertainties in this conversion: the quark-mass ratio of Eq. (5) (reflecting mainly the uncertainty of $m_s \approx 150$ to 200 MeV), and the often quoted error in the ρ-parameter fit[17] of the low-energy ν and $\bar{\nu}$ neutral-current data. (Namely, $\rho = 1.02 \pm 0.02$ may be a bit too optimistic.[18]) For definiteness we shall in the following take the quark-mass ratio to be 8 and allow for $\rho - 1$ to be as large as 0.06 to obtain

$$m_o \lesssim 480 \text{ GeV}, \quad m_h \lesssim 60 \text{ GeV}. \tag{10}$$

Although they are supposed to be upper limits, the actual quark-mass values are likely to saturate them. The argument for this expectation is based on the observation that a number of simple grand unification models predict that $m_h/m_\kappa \simeq 2$ to 3 (recall the successful relation[19] between m_τ and m_b) and the present lower limit on m_κ is already 22 GeV.[20] Also, although we do not really understand the quark-mass hierarchy, the empirical rule that masses in each succeeding generation increase by a factor of λ^{-2} would indicate that the bounds in (10) are saturated. Now we will apply the previous simple model to the four-generation case, the Fritzsch *Ansatz* for the mass matrices and mass hierarchy of Eq. (3) being extended to

$$m_h{:}m_b{:}m_s{:}m_d = 1{:}\lambda^2{:}\lambda^4{:}\lambda^6,$$

$$m_o{:}m_t{:}m_c{:}m_u = 1{:}\lambda^2{:}\lambda^4{:}\lambda^8.$$

Besides Eqs. (6a) and (6b), we also have

$$(m_b/m_h)^{1/2} - (m_t/m_o)^{1/2} = B\lambda^2, \tag{11}$$

with B being $O(1)$. The orthogonal matrix of Eq. (8) will have the extra matrix elements of $O_{i4} = (m_i/m_4)^{1/2}$, $O_{43} = -(m_3/m_4)^{1/2}$; and O_{42}, O_{41} which are of $O(\lambda^4)$ or higher can be neglected. Our experience for the three-generation case also leads us to the phase choice of $\beta_i = \gamma_i$ (1,..., 4), the same value (or an extra 180° to reverse the sign) for all α_i's, and the mass-matrix phase structure $\gamma_2 = \gamma_3 = \gamma_4 = \gamma_1 + \pi/2$.

For the elements involving the fourth-generation quarks, we have, besides $U_{oh} = O(1)$,

$$\begin{aligned} &U_{th} = -U_{ob} = B\lambda^2, \quad U_{ch} = -U_{os} = -Ba\lambda^3, \\ &U_{uh} = iB\xi\lambda^4, \quad U_{od} = B\lambda^4(-a+i\xi), \end{aligned} \tag{12}$$

where we have introduced the new notations $a\lambda = (m_s/m_b)^{1/2}$ and $\xi\lambda^2 = (m_u/m_t)^{1/2}$. Thus we have $O(1)$ parameters A, B, and a, and $O(\lambda)$ parameters σ, η, and ξ. The magnitudes of the fourth-generation U_{ij} elements are summarized below:

$$U = \begin{pmatrix} 1 & \lambda & \lambda^4 & \lambda^5 \\ \lambda & 1 & \lambda^2 & \lambda^3 \\ \lambda^3 & \lambda^2 & 1 & \lambda^2 \\ \lambda^4 & \lambda^3 & \lambda^2 & 1 \end{pmatrix}. \tag{13}$$

Based on the observation that each of the mass ratios in U_{th} [Eqs. (11) and (12)] is larger than the corresponding ones in U_{cb} [Eq. (9)] we expect that the coefficient B should be somewhat greater than the coefficient A, thus $U_{th} \gtrsim U_{cb}$. This result differs markedly from the often stated expectation of progressive suppression of mixings between neighboring generations as quarks get heavier.[1] In fact, our model suggests that all mixings between neighbors are of $O(\lambda^2)$ except the Cabibbo angle which, in view of this analysis, is "anomalously large" reflecting the fact the m_u is "anomalously small," breaking the general pattern of mass hierarchy.

Finally, we have also studied the standard-model calculation of the kaon CP-impurity parameter ϵ in the presence of the fourth generation. ϵ is dominated by the short-distance physics as represented by the "box diagram":

$$\epsilon \propto B(i,j) = E(i,j)\,\mathrm{Im}(U_{is}^* U_{id} U_{js}^* U_{jd}).$$

Namely, the contribution of the box diagram with internal quark lines $i,j = c$, t, and o is a product of two competing factors: a kinematical factor $E(i,j)$ which increases with quark masses and the (imaginary part of) mixing angle factor which decreases for heavy quarks. Since U_{os} and U_{od} in (12) and (13) are not extremely small, one may think that the fourth-generation contribution will be of major significance. Our detailed calculation[8] shows that this is not the case. For the dominant term $B(c,t)$ being normalized to $O(1)$, we have the following subleading contributions: $B(c,c)$, $B(t,t)$ are $O(\lambda)$, and $B(c,o)$, $B(t,o)$ are $O(\lambda^2)$, and $B(o,o)$ is $O(\lambda^3)$. Similarly, one can show that the fourth-generation contribution to ϵ' will not be of importance if such nonleptonic weak decays are correctly described by the "penguin diagram."

In summary, we have argued that existing data on quark masses and mixing angles are consistent with the assumptions that the charge $\frac{2}{3}$ and $-\frac{1}{3}$ quark mass matrices have the Fritzsch form and are closely proportional to each other. The ratio of the up and down quarks in each generation is approximately the same. This pattern is broken mainly in the light-quark sector (an "anomalously small" m_u, and "maximal CP phase" associated with the first generation). Wolfenstein parameters of the KM matrix are computed with the prediction that $\Gamma(b\to ue\nu)/\Gamma(b\to ce\nu)\simeq 0.6\%$ and the case for $m_t\simeq 40$ GeV is strengthened. If the fourth-generation fermions exist, we expect $m_h\simeq 60$ GeV and m_o in the 400–500-GeV range. They cannot be higher because of the electroweak ρ-parameter constraint. In fact if the fourth generation indeed exists, we would anticipate that when M_W and M_Z are measured precisely they will show a deviation from $\rho=1$ larger than the limit of 0.04 as deduced at present from the low-energy neutrino neutral-current data. The octa- and hepta-quark masses (m_o,m_h) probably are not lower than the above values because of the pattern of mass hierarchy and because it is likely that $m_h/m_\kappa\simeq 2$ to 3. Thus if this fourth generation exists we would expect $m_\kappa\simeq 20$–30 GeV and it should be produced copiously in the KEK e^-e^+ collider TRISTAN. In short, we conclude that the systematics of the quark mass matrices suggests that the fourth generation of fermions should be "just around the corner" or, equivalently, it is on the verge of being ruled out.

This work was initiated while we were visitors at Brookhaven National Laboratory; the hospitality of the theoretical physics group there is gratefully acknowledged. Our research was supported in part by the National Science Foundation and by the Department of Energy.

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