Random-Field Critical Behavior: Finite-Size Effects

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We present the results of the first analytical study of finite-size effects in a random-field model. In particular we consider the random-field spherical model in partially infinite geometries (infinite in d' dimensions and finite in $d-d'$ dimensions) under periodic and antiperiodic boundary conditions in the finite directions. We find a simultaneous dimensional reduction by 2 in d and d' for $4<d<6$ and $2\leqslant d'\leqslant 4$. For $0\leqslant d'<2$, we find new power laws describing the approach to bulk in different geometries.

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Recently, there has been considerable interest in the effects of quenched random fields on the bulk critical behavior of n-vector models.[1] For O(n) models with $n>2$, it is commonly believed that the critical behavior of a random-field model (RFM) in d dimensions is the same as that of the pure model in (reduced) $d-2$ dimensions.[2] For example, the lower and upper critical dimensions of the RFM are believed to be 4 and 6, respectively. On the other hand, very little is known analytically about finite-size effects in RFM. This is in contrast to pure systems where the finite-size scaling theory[3] and the approaches to bulk for $T>T_c$ and $T<T_c$ are well studied.[4,5]

The aim of this Letter is twofold. First, we shall solve analytically the spherical random-field model in various finite-size geometries under periodic and antiperiodic boundary conditions (without using the replica trick). Second, we shall show that, in the first-order transition region, i.e., $H=0$, $T<T_c$, the approach to bulk of various quantities is mostly by power-law exponents which are related to those of the pure case by dimensional reduction by 2, not only in d, but also in d' (the number of dimensions in which the system is infinite), for $2\leqslant d'<4$. For $0\leqslant d'<2$, the approach is by exponents which are unrelated to those of the pure system and for $d'=4$, which is the lower critical dimensionality, the approach is exponentially fast for all quantities except for the free energy.

Our results are summarized in Table I for different geometries. The dimensional reduction can be seen by comparing the results for a (d,d')-dimensional RFM system with that of a $(d-2,\ d'-2)$-dimensional pure system for $2\leqslant d'\leqslant 4$. For $0\leqslant d'<2$, the results cannot be understood by dimensional reduction. But, for the case $d'=0$, the "block" geometry, they can be understood by an appeal to random-walk arguments leading to an exponent $d/2$ in contrast to the exponent d in the pure system.[4] For the free-energy density in the antiperiodic case, the approach to bulk is like L^{-2} for $H=0$, $T<T_c$ in all geometries (here L is the length of the system in each of the finite directions). This can be understood as a result of the presence of a helicity modulus[6] in the low-temperature phase.

Even though we have derived these results only for the spherical model, by previous experience with pure systems,[4,5] it is tempting to conjecture that these exponents may be correct for all n-vector O($n>2$) models. In the rest of the Letter, we present a sum-

TABLE I. Approach to bulk of the free energy f, susceptibility χ, specific heat $C^{(S)}$ (all per unit volume) for $T<T_c$, $H=0$, $L\to\infty$ for the RFM and pure system for periodic (PBC) and antiperiodic (APBC) boundary conditions. The geometry of the system is $L^{d-d'}\times\infty^{d'}$, $4<d<6$, $0\leqslant d'\leqslant 4$ for the RFM and $2<d<4$, $0\leqslant d'\leqslant 2$ for the pure system.

	f(PBC)		f(APBC)		χ(PBC or APBC)		$C^{(S)}$(PBC or APBC)	
d'	RFM	Pure	RFM	Pure	RFM	Pure	RFM	Pure
0	$L^{-d/2}$	$L^{-d}\ln L^{-1}$	L^{-2}	L^{-2}	$L^{d/2}$	L^{d}	$L^{-d/2}$	L^{-d}
1	$L^{-(2/3)(d-1)}$	L^{-d}	L^{-2}	L^{-2}	$L^{2/3(d-1)}$	$L^{2(d-1)}$	$L^{-(2/3)(d-1)}$	$L^{-2(d-1)}$
2	$L^{-(d-2)}\ln L^{-1}$	L^{-d}	L^{-2}	L^{-2}	L^{d-2}	$\exp(c_2L)$	$L^{-(d-2)}$	$\exp(-c_2L)$
3	$L^{-(d-2)}$	. . .	L^{-2}	. . .	$L^{2(d-3)}$	. . .	$L^{-2(d-3)}$	. . .
4	$L^{-(d-2)}$	. . .	L^{-2}	. . .	$\exp(c_1L)$	. . .	$\exp(-c_1L)$	. . .

mary of the techniques and our principal results.[7]

We consider a system of N spins located at sites $\mathbf{r}_i$ of a hypercubical lattice of size $(N_1 a \times N_2 a \times \cdots \times N_d a = N a^d)$ interacting via the nearest-neighbor Hamiltonian,[8]

$$H = -J \sum_{\mathrm{nn}} S_i S_j - \sum_{i=1}^{N} H_i S_i + \lambda \sum_{i=1}^{N} S_i^2, \tag{1}$$

in the usual notation, where, in addition, H_i are independent, randomly distributed fields such that

$$\overline{H_i} = H, \quad \text{all } i, \tag{2a}$$

$$\overline{H_i H_j} = H^2 + \sigma^2 \delta_{ij}, \quad \text{all } i,j. \tag{2b}$$

Here the bar denotes the average with respect to the probability distribution. The spherical field λ has been introduced so as to satisfy the mean spherical constraint,

$$\sum_{i=1}^{N} \overline{\langle S_i^2 \rangle} = N, \tag{3}$$

where the angular brackets denote the usual thermal average. By using standard techniques[8,9] we get for the free energy per spin

$$F(\beta, H_i, \lambda; N) = \frac{1}{2\beta N} \sum_{\mathbf{k}} \ln[\beta(\lambda - \mu_{\mathbf{k}})] - \frac{1}{4N} \sum_{\mathbf{k}} \frac{|H_{\mathbf{k}}|^2}{(\lambda - \mu_{\mathbf{k}})}, \tag{4}$$

where $\beta = T^{-1}$ (we take $k_{\mathrm{B}} = 1$), and the eigenvalues $\mu_{\mathbf{k}}$ are given by

$$\mu_{\mathbf{k}} = 2J \sum_{j=1}^{d} \cos k_j, \quad k_j = 2\pi(n_j + \tau)/N_j,$$

$$n_j = 0, 1, 2, \ldots, (N_j - 1). \tag{5}$$

Here $H_{\mathbf{k}} = \sum_{i=1}^{N} H_i y_{\mathbf{k}}(\mathbf{r})$ where the eigenfunctions are given by

$$y_{\mathbf{k}}(\mathbf{r}) = N^{-1/2} \exp(i \mathbf{k} \cdot \mathbf{r}) \tag{6}$$

and $\tau = 0$ $(\frac{1}{2})$ for periodic (antiperiodic) boundary conditions. Taking the average of (4) with respect to the distribution with the use of (2), we get, for $H = 0$,

$$\bar{F} = \frac{1}{2\beta N} \sum_{\{n_j\}} \ln\left[\beta\left(\lambda - 2J \sum_{j=1}^{d} \cos k_j\right)\right] - \frac{\sigma^2}{4N} \sum_{\{n_j\}} \left[\lambda - 2J \sum_{j=1}^{d} \cos k_j\right\}^{-1}. \tag{7}$$

The constraint equation becomes

$$2K = \frac{1}{N} \sum_{\{n_j\}} \left(\frac{\lambda}{J} - 2 \sum_{j=1}^{d} \cos k_j\right]^{-1} + \frac{K}{2N} \left(\frac{\sigma}{J}\right)^2 \sum_{\{n_j\}} \left[\frac{\lambda}{J} - 2 \sum_{j=1}^{d} \cos k_j\right]^{-2}. \tag{8}$$

The terms for $\sigma = 0$ in these equations are the pure-system ones studied by Singh and Pathria[5] in detail for $2 < d < 4$. Their methods can easily be extended for $4 < d < 6$, for the case at hand. The terms with $\sigma \neq 0$ are related to the $\sigma = 0$ terms by differentiation with respect to λ. Because of this simple structure of Eqs. (7) and (8) we were able to study the RFM in detail in the general geometry $L^{d-d'} \times \infty^{d'}$. In principle, the method amounts to replacing sums in d' infinite directions by integrals and studying the remaining sum in $d^* = d - d'$ dimensions by using the Poisson summation formula.

We have studied the free energy, the specific heat, and the susceptibility (with $H = 0$; $\sigma \neq 0$) using these methods. As an example, we quote the result for the singular part of the specific heat. We get

$$C^{(S)} = 1024 \pi^{d/2} T_c(\sigma) \left[\frac{J}{\sigma T_c(0)}\right]^2 \frac{(ay/L)^{6-d}}{[\Gamma((6-d)/2) + 2K((d-6)/2 | d^*; y; \tau)]}, \tag{9}$$

where the parameter y is given by the implicit equation

$$-\left[\frac{T - T_c(\sigma)}{T_c(0)}\right]\left(\frac{J}{\sigma}\right)^2 = \frac{(ay/L)^{d-4}}{64\pi^{d/2}} \left[\Gamma\left(\frac{4-d}{2}\right) + 2K\left(\frac{d-4}{2}\middle| d^*; y; \tau\right)\right]. \tag{10}$$

Here $y = \frac{1}{2}(L/a)(\lambda - 2dJ)^{1/2} = L/2\xi$, ξ is the correlation length,

$$T_c(\sigma) = T_c(0)[1 - \tfrac{1}{4}(\sigma/J)^2 I_d], \tag{11}$$

$$I_d = \frac{1}{4(2\pi)^d} \int_0^{2\pi} d\theta_1 \cdots \int_0^{2\pi} d\theta_d \left[\sum_{j=1}^{d} (1 - \cos\theta_j)\right]^{-2}, \tag{12}$$

and, finally,

$$K(z|d^*;y;\tau)=\sum_{\mathbf{q}(d^*)}\left[\prod_{j=1}^{d^*}\cos(2\pi q_j\tau)\right]\frac{K_z(2|\mathbf{q}|y)}{(|\mathbf{q}|y)^z},\quad |\mathbf{q}|\neq 0, \tag{13}$$

where $K_z(x)$ is the modified Bessel function.

The various properties of these K sums have been studied in detail previously.[5] For $T > T_c$, as $L \to \infty$, $y \to \infty$ and the K sums vanish as e^{-2y}, so that $C^{(S)}$ approaches its bulk limit as $e^{-L/\xi}$. For $T < T_c$, $L \to \infty$, $y^2 \to 0$ $(-d^*\pi^2/4)$ for periodic (antiperiodic) boundary conditions and the K sums diverge. Using their properties,[5] we find that the bulk limit is approached as $L^{-\omega}$, where

$$\omega=2(d-d')/(4-d'),\quad 0\leq d'<4, \tag{14}$$

and exponentially fast only for $d'=4$. Comparing ω with the rate of approach in the pure case,[4,5] viz.,

$$\zeta=2(d-d')/(2-d'),\quad 0\leq d'<2, \tag{15}$$

we find dimensional reduction by 2 in both *d and d'*. One can similarly obtain all the other results mentioned in Table I.

We mention only that in the antiperiodic case, the free energy per unit volume deviates from the bulk for $T < T_c$, $L \to \infty$ by $\frac{1}{2}d^*\pi^2\Upsilon(T)/L^2$ where $\Upsilon(T)$, the helicity modulus,[6] is given by

$$\Upsilon(T)=(2J/a^{d-2})[M_0(T)]^2, \tag{16}$$

in terms of the spontaneous magnetization (bulk)

$$[M_0(T)]^2=\frac{T_c(\sigma)-T}{T_c(0)},\quad T\leq T_c(\sigma). \tag{17}$$

As shown in Table I, the analogous calculation with periodic boundary conditions yields a power law $L^{-\epsilon}$ with $\epsilon > 2$. The relation (16) is identical to the one found in the pure case,[5,8] although $M_0(T)$ shows the effects of the random field by being only partially saturated {i.e., $M_0(0)=[T_c(\sigma)/T_c(0)]^{1/2}<1$}.

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