Subalgebras of Loop Algebras and Symmetries of the Kadomtsev-Petviashvili Equation

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(Received 29 July 1985)

It is shown that the symmetry algebra of the Kadomtsev-Petviashvili equation can be related to an infinite-dimensional subalgebra of the loop algebra $[\mathrm{SL}(5,R) \otimes R(t,t^{-1}) \oplus [R(t,t^{-1}) d/dt]$. The algebra is used to generate new classes of solutions of the Kadomtsev-Petviashvili equation, depending on several arbitrary functions.

PACS numbers: 02.20.+b, 03.40.Kf

Infinite-dimensional Lie groups and Lie algebras, particularly Kac-Moody algebras and loop algebras,[1] are playing an ever increasing role in contemporary physics (for recent reviews and summaries see, e.g., Kac,[1] Dolan,[2] Jimbo and Miwa,[3] and Drinfeld and Sokolov[4]). In connection with completely integrable Hamiltonian systems they have been used both to generate such systems and to integrate them.

The purpose of this Letter is to point out that infinite-dimensional subalgebras of affine loop algebras also occur in a different and seemingly independent way in the study of certain integrable nonlinear partial differential equations in 2+1 dimensions. Namely, they play the role of symmetry algebras of such important equations as the Kadomtsev-Petviashvili (KP) equation,[5] the Davey-Stewartson equation,[6] and other equations in more than one spatial dimension.[7] In this context we use the words "symmetry algebra" in the most restricted and classical sense, namely, that of the Lie algebra of a Lie group of point transformations, leaving a given equation or system of equations invariant.[8,9] Such a Lie group transforms solutions of the system among each other and can be used to derive new solutions by the method of symmetry reduction.

In this Letter we concentrate on the Kadomtsev-Petviashvili equation (the "two-dimensional Korteweg–de Vries equation")

$$\Omega^\sigma(t,x,y;u) \equiv [u_t + \tfrac{3}{2} u u_x + \tfrac{1}{4} u_{xxx}]_x + \tfrac{3}{4}\sigma u_{yy} = 0, \quad \sigma = \pm 1 \tag{1}$$

and present its symmetry algebra, depending on three arbitrary functions of time t. We relate this algebra to an infinite-dimensional subalgebra of the simple loop algebra $A_4^{(1)}$, classify its one-dimensional subalgebras, and use these to generate new solutions. We concentrate on the results only; details of mathematical nature and proofs will be published elsewhere.[10]

As physical motivation let us mention that the KP equation describes the propagation of two-dimensional waves on the surface of a fluid or a plasma, as well as internal waves on the interface of two fluids (e.g., two layers of water of different densities). The mathematical motivation stems from the fact that this equation is, in a well-defined sense, generic among integrable nonlinear differential equations, especially those in more than one spatial dimension.[3]

The symmetry algebra of the KP equation consists of differential operators of the form

$$V = \tau(t,x,y;u)\partial_t + \xi(t,x,y;u)\partial_x + \eta(t,x,y;u)\partial_y + \phi(t,x,y;u)\partial_u, \tag{2}$$

such that their fourth prolongation[11] (since the equation is of order four) satisfies

$$\mathrm{pr}^4 V \circ \Omega^\sigma(t,x,y;u)|_{\Omega^\sigma(t,x,y;u)=0} = 0.$$

This condition is imposed by application of the differential operator $\mathrm{pr}^4 V$ to Ω^σ and then elimination, in as much as possible, of all y derivatives of u by use of the KP equation and its differential consequences. Equating to zero the coefficients of linearly independent expressions in the t and x derivatives of u, we obtain a system of "determining equations" for the coefficients τ, ξ, η, and ϕ in (2). The procedure is standard and can be implemented on a computer, with use of symbolic manipulation languages such as REDUCE[12] or MACSYMA.[7] The result, first obtained by Schwarz,[13] is that a general element of the symmetry algebra of the KP equation has the form

$$V = X(f) + Y(g) + Z(h), \quad X(f) = f\partial_t + [\tfrac{1}{3}xf' - \tfrac{2}{9}\sigma y^2 f'']\partial_x + \tfrac{2}{3}yf'\partial_y - [(4\sigma/27)y^2 f''' - \tfrac{2}{9}xf'' + \tfrac{2}{3}uf']\partial_u,$$

$$Y(g) = g\partial_y - \tfrac{2}{3}\sigma y g'\partial_x - \tfrac{4}{9}\sigma y g''\partial_u, \quad Z(h) = h\partial_x + \tfrac{2}{3}h'\partial_u. \tag{3}$$

Here $f(t)$, $g(t)$, and $h(t)$ are arbitrary functions of time of class C^∞ and the primes denote time derivatives. We thus see that this Lie algebra is indeed infinite dimensional as its elements are labeled by these arbitrary functions.

Let us first show that (3) contains a structure that is a subalgebra of an algebra obtained by adding a derivation algebra, with basis $d_s = t^{s+1} d/dt$, to a loop algebra. To see this, let us follow a procedure similar to the one described in Ref. 1, Chap. 7, providing a realization of nontwisted affine Lie algebras. Instead of starting from a finite simple Lie algebra, however, we start from a solvable one, namely, the algebra L_0 given by the following vector fields in three variables:

$$\Delta = x\,\partial_x + 2y\,\partial_y - 2u\,\partial_u, \quad Q = y\,\partial_x, \quad Y = \partial_y,$$
$$X = \partial_x, \quad A = -\sigma y^2\,\partial_x + x\,\partial_u, \quad S = y\,\partial_u,$$
$$P = y^2\,\partial_u, \quad U = \partial_u.$$

The algebra L_0 is solvable, its nilradical is $\{Y,A,P,Q,X,S,U\}$, and it contains a five-dimensional Abelian ideal $\{P,Q,X,S,U\}$. The lowest-dimensional simple real Lie algebra that contains a five-dimensional Abelian subalgebra is SL(5,R) and indeed L_0 can be identified with a subalgebra of SL(5,R) realized by the following matrices:

$$\xi = \begin{bmatrix} \delta & -a & \sigma p & s & u \\ 0 & 0 & -a & q & x \\ 0 & 0 & -\delta & -2\sigma y & 0 \\ 0 & 0 & 0 & -\delta/3 & -y \\ 0 & 0 & 0 & 0 & \delta/3 \end{bmatrix}$$

(e.g., the matrix representing A is obtained by setting $a=1$ and all other entries equal to 0). A natural grading is provided by attribution of a degree μ $(0 \le \mu \le 4)$ to each element of L_0, where μ is the distance of the corresponding element in the matrix ξ from the diagonal (e.g., $\mu=0$ for Δ $\mu=4$ for U). Now let $L = R(t,t^{-1})$ be the algebra of real Laurent *polynomials* in t and $D = R(t,t^{-1})\,d/dt$ be a (simple) algebra of derivations. Consider a subalgebra of the loop algebra $A_4^{(1)}$, extended by D, generated by the elements

$$X(t^n) = \tfrac{1}{3} n t^{n-1}\Delta + \tfrac{2}{9} n(n-1) t^{n-2} A - (4\sigma/27) n(n-1)(n-2) t^{n-3} P + t^n \partial_t,$$
$$Y(t^n) = t^n Y - \tfrac{2}{3}\sigma n t^{n-1} Q - (4\sigma/9) n(n-1) t^{n-2} S, \quad Z(t^n) = t^n X + \tfrac{2}{3} n t^{n-1} U. \tag{4}$$

If we attribute the degree n to a monomial t^n, we find that $X(t^n)$, $Y(t^n)$, and $Z(t^n)$ have degrees $n-1$, $n+1$, and $n+3$, respectively. We see that the subalgebra (4) of $A_4^{(1)} \oplus D$ coincides with the algebra of vector fields (3), once the functions $f(t)$, $g(t)$, and $h(t)$ are restricted to Laurent polynomials.

We emphasize the difference between the realization of the loop algebra constructed above and that of other applications. The Laurent polynomials in our case are functions of time t; in the other cases (Refs. 1–4 and many others) the corresponding expansions are done in terms of a (complex) variable λ, related to a spectral parameter in a Lax pair and in an inverse scattering problem. The physical applications, as we will see below, are also completely different.

The KP algebra (3) allows a Levi decomposition

$$L = S \oplus N, \quad S = \{X(f)\}, \quad N = \{Y(g), Z(h)\},$$

where S is an infinite-dimensional simple Lie algebra (isomorphic to the algebra of real vector fields on S^1, simple according to a proof by Cartan[14]) and N is a nilpotent ideal in L. For a discussion of the relation between the algebra of vector fields on S^1 and the Virasoro algebra see the work of Goodman and Wallach.[15]

The "physically obvious" symmetries of the KP equation are obtained by restricting the functions in (3) to first-order polynomials. We obtain the translations $X(1) = \partial_t$, $Y(1) = \partial_y$, and $Z(1) = \partial_x$, the dilation $X(t) = t\,\partial_t + \frac{1}{3}x\,\partial_x + \frac{2}{3}y\,\partial_y - \frac{2}{3}u\,\partial_u$, the "quasirotation" $Y(t) = t\,\partial_y - \frac{2}{3}\sigma y\,\partial_x$, and the Galilei boost $Z(t) = t\,\partial_x + \frac{2}{3}\partial_u$. Including a second-order polynomial we obtain an SL(2,R) subalgebra $\{X(1), X(t), X(t^2)\}$, where $X(t^2)$ generates a conformal-type transformation.

In order to generate solutions of the KP equation systematically by symmetry reduction we first need to classify the low-dimensional subalgebras of the KP algebra into conjugacy classes under the adjoint action of the corresponding Lie group. This is done elsewhere[10] for dimensions $n=1$, 2, and 3. Here we summarize the result for $n=1$ only, namely, an arbitrary element of the KP algebra can be conjugated into $X(1)$ [if $f(t)\neq 0$], $Y(1)$ [if $f(t)=0$, $g(t)\neq 0$], or $Z(1)$ [if $f(t)=0$, $g(t)=0$, and $h(t)\neq 0$].

Symmetry reduction is performed in a standard manner, namely, we take an element X of the symmetry algebra and write the first-order linear partial diffential equation $XI(t,x,y;u)=0$. Solving the corresponding characteristic equations we obtain two symmetry variables $\xi(t,x,y)$, $\eta(t,x,y)$, and an expression for the solution of the KP equation:

$$u(t,x,y) = \alpha(t,x,y)\,q(\xi,\eta) + \beta(t,x,y), \tag{5}$$

where ξ, η, α, and β are explictly given. The function $q(\xi,\eta)$ is subject to a partial differential equation in the *two* variables ξ and η, obtained by substitution of (5) into the KP equation (1).

Let us consider each class of elements separately.

(1) $f(t)\neq 0$.—Following the above procedure we obtain solutions of the KP equation depending on three

arbitrary functions $f(t)$, $g(t)$, and $h(t)$:

$$u(t,x,y) = f^{-2/3} q(\xi,\eta) + \frac{2f'}{9f}x + \frac{4\sigma}{27}\,\frac{(2f'g - 3fg')y}{f^2} + \frac{4\sigma}{81}\,\frac{2f'^2 - 3ff''}{f^2}\,y^2 + \frac{2\sigma g^2}{9f^2} + \frac{2h}{3f},$$
$$\xi = \left[x + \frac{2\sigma}{3}\,\frac{gy}{f} + \frac{2\sigma}{9}\,\frac{f'y^2}{f}\right] f^{-1/3} - \int_0^t \left[\frac{2\sigma}{3} g^2(s) f^{-7/3}(s) + h(s) f^{-4.3}(s)\right] ds, \quad \eta = y f^{-2/3} - \int_0^t g(s) f^{-5/3}(s)\, ds. \tag{6}$$

The function $q(\xi,\eta)$ must satisfy the Boussinesq equation

$$\sigma q_{\eta\eta} + (q^2)_{\xi\xi} + \tfrac{1}{3} q_{\xi\xi\xi\xi} = 0.$$

(2) $f(t) = 0$, $g(t) \neq 0$.—We obtain solutions of the KP equation depending on two arbitrary functions $g(t)$ and $h(t)$:

$$u(t,x,y) = \frac{1}{\sqrt{g}} q(\xi,\eta) + \frac{g'x}{3g} + \frac{y(2gh' - g'h)}{3g^2} + \frac{\sigma(g'^2 - 2gg'')y^2}{9g^2} - \frac{\sigma h^2}{2g^2},$$
$$\xi = \frac{1}{\sqrt{g}}\left[x - \frac{hy}{g} + \frac{\sigma g' y^2}{3g}\right], \quad \eta = \int_0^t g^{-3/2}(s)\, ds, \tag{7}$$

where $q(\xi,\eta)$ satisfies a (differentiated) Korteweg–de Vries equation

$$[q_\eta + \tfrac{3}{2} q q_\xi + \tfrac{1}{4} q_{\xi\xi\xi}]_\xi = 0.$$

(3) $f(t) = g(t) = 0$, $h(t) \neq 0$.—We obtain a linear equation that can be solved directly to prove the following explicit solution of the KP equation:

$$u(t,x,y) = \frac{2h'(t)x}{3h(t)} - \frac{4\sigma h''(t)y^2}{9h(t)} + r(t)y + s(t), \tag{8}$$

where $r(t)$ and $s(t)$ are arbitrary functions.

To summarize, formula (8) gives an explicit solution of the KP equation. Formulas (6) and (7) allow us to "boost" arbitrary solutions of the Boussinesq or Korteweg–de Vries equations into solutions of the KP equation.

As final comments let us mention that the symmetry algebras of the modified KP equation[7] and of the Davey-Stewartson equation[16] have very similar properties to those of the KP algebra. It appears that a combination of the occurrence of Kac-Moody algebras, the development of systematic methods of subgroup classification, and the possibilities offered by symbolic manipulation on computers should greatly enhance the usefulness of the method of symmetry reduction for nonlinear partial differential equations.

This work was supported by the Natural Sciences and Engineering Research Council of Canada and the Fonds pour la Formation de Chercheurs et l'Aide à la Recherche, pour l'aide et le soutien á la Recherche du Gouvernement du Québec.

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