One-Loop Finiteness in O(32) Open-Superstring Theory

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It is shown that the one-loop infinities cancel between diagrams for $M=5$ open external lines, and any M for suitably constrained kinematics, in the O(32) superstring theory when one employs a principal-part prescription for regularization.

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Superstring theory offers the first real hope of a successful marriage between quantum mechanics and general relativity. The theory may lead to controllable quantum corrections to the classical Einstein theory. Already, chiral-anomaly cancellation has singled out a particular allowed gauge group O(2) for open superstrings,[1] the gauge and gravity anomaly cancellations arising from a series of apparently miraculous numerical coincidences. Further, and even more interestingly, the one-loop graphs are finite for $M=4$ external lines[2] provided one adopts a principal-part (PP) prescription for regularization. If this finiteness generalizes to all $M>4$ (including $M=6$) the absence of anomalies will be explained.

We shall do the calculation for gauge group O(N) and show how finiteness holds for $M=5$ external lines, and any M for suitably restricted kinematics, only for $N=32$. For general M there are two divergent one-loop diagrams with open external strings: the annulus (an orientable planar diagram) and the Moebius strip (nonorientable). These two infinities must cancel if the theory is finite.

For the case $M=4$ discussed in Ref. 2 the annulus and Moebius strip have amplitudes A_{4P} and A_{4N} of the respective forms

$$A_{4P}+A_{4N}=NK_4\int_0^1\frac{dq}{q}F_4(q^2)-8K_4\int_0^1\frac{dq}{q}F_4(-\sqrt{q}\,). \tag{1}$$

The PP prescription is to use integration variables $\lambda=q^2$ in A_{4P} and $\lambda=\sqrt{q}$ in A_{4N}, allowing us to rewrite for $N=32$

$$A_{4P}+A_{4N}=16K_4\,\mathrm{P}\int_{-1}^{+1}\frac{d\lambda}{\lambda}F_4(\lambda), \tag{2}$$

where P means principal part. There is a real ambiguity here since if we had put (say) $\lambda=q$ rather than $\lambda=q^2$ in A_{4P} the result would be infinite but (a) we know that only $N=32$ is consistent because of anomaly cancellation[1] and (b) the PP prescription singles out $N=32$. In the absence of a detailed regularization procedure it seems reasonable to adopt this PP prescription as a working hypothesis.

The crucial factor 8 in Eq. (1) arises from the doubled integration region of the three ν_I variables (throughout we employ notation from the review article by Schwarz[3]). Very naively, one expects a factor 2^{M-1} here for general M, so that unless the superstring algebra leads to nontrivial factors of 2 which precisely compensate this the finiteness will fail and the open superstring could be eliminated. To anticipate our result, the required nontrivial factor 2^{4-M} *is* generated by a "superstring miracle" in the loop calculation.

We work in the light-cone gauge where the vertex for emission of a massless gauge boson (with $k^+=0$ and $\zeta^+=0$) is[3]

$$V(k)=\zeta^i(P^i+R^{ij}k^j)\,V_0(k). \tag{3}$$

Since vertices with $k^+\neq 0$ are rather unwieldy, we will restrict ourselves to the $k^+=0$ case. This restriction allows the discussion of $M\leqslant 10$ external spinless, massless particles. For all external particles of spin 1 (with $\zeta^+=0$) the restriction is stronger: $M\leqslant 5$ for a parity-nonconserving amplitude and $M \leqslant 7$ for a parity-conserving amplitude. Later on when we discuss the general M case the result is understood to be for suitably constrained kinematics. We shall first consider the pentagon diagram ($M=5$) because it already contains one nontrivial factor of 2 and it will then be shown how these factors generalize to all M. The $M=4$ case has only one nonvanishing piece (R_0^4) being close to the $M=2$ and 3 cases which completely vanish because of a superstring nonrenormalization theorem; this is why the general result was not obvious from Ref. 2.

In the pentagon, we identify four nonvanishing contributions as coming from the combinations

$$R_0^5, \tag{4a}$$

$$R_0^3 (S_n S_0)^2, \tag{4b}$$

$$R_0^4 P(n=0), \tag{4c}$$

$$R_0^4 P(n \neq 0), \tag{4d}$$

which have (a) ten R_0 factors or (b), (c), (d) eight R_0 factors, the minimum possible; contribution (a) is pure fermion zero modes, (b) involves only fermions but includes nonzero modes [note that $R_0^4 (S_n)^2$ vanishes because there is no second-rank antisymmetric tensor in O(8)], (c) is pure zero modes but includes bosons, and (d) involves bosonic nonzero modes.

The result for the annulus is (external momenta and polarization vectors are $k_I^{j_I}$ and $\zeta_I^{i_I}$, $1 \leqslant I \leqslant 5$)

$$A_{5P} = g^5 \operatorname{Tr}(\Lambda^a \Lambda^b \Lambda^c \Lambda^d \Lambda^e) N \int_0^1 \left(\prod_{I=1}^{5} dx_I \right) W^{-1} \left(-\frac{2\pi}{\ln W} \right)^5 \prod_{1 \leqslant I \leqslant J \leqslant 5} (\psi_{IJ})^{k_I k_J} [I_a + I_b + I_c + I_d], \tag{5}$$

where $I_{a,b,c,d}$ correspond to the four nonvanishing pieces, (a) through (d). The detailed expressions are

$$I_a = \zeta_1^{i_1} \zeta_2^{i_2} \zeta_3^{i_3} \zeta_4^{i_4} \zeta_5^{i_5} k_1^{j_1} k_2^{j_2} k_3^{j_3} k_4^{j_4} k_5^{j_5} \operatorname{Tr}(R_0^{i_1 j_1} R_0^{i_2 j_2} R_0^{i_3 j_3} R_0^{i_4 j_4} R_0^{i_5 j_5}), \tag{6}$$

$$I_b = \zeta_1^{i_1} \cdots k_5^{j_5} \frac{1}{64} \sum_{l=1}^{\infty} \frac{1}{1-W^l} \left\{ \left[\left(\frac{W}{x_2} \right)^l - x_2^l \right] t^{i_1 j_1 i_2 j_2 ij} \operatorname{Tr}(R_0^{ij} R_0^{i_3 j_3} R_0^{i_4 j_4} R_0^{i_5 j_5}) \right.$$
$$\left. + \left[\left(\frac{W}{x_2 x_3} \right)^l - (x_2 x_3)^l \right] t^{i_1 j_1 i_3 j_3 ij} \operatorname{Tr}(R_0^{ij} R_0^{i_2 j_2} R_0^{i_4 j_4} R_0^{i_5 j_5}) + \text{cyclic permutations} \right\}, \tag{7}$$

where

$$t^{i_1 j_1 i_2 j_2 i_3 j_3} = \operatorname{Tr}(\gamma^{i_1 j_1} \gamma^{i_2 j_2} \gamma^{i_3 j_3}) = 32 [\delta_{i_1 j_2} \delta_{i_2 j_3} \delta_{i_3 j_1} - \delta_{i_1 i_3} \delta_{i_2 j_3} \delta_{j_1 j_2} - \delta_{i_1 i_2} \delta_{j_2 j_3} \delta_{i_3 j_1} + \delta_{i_1 i_3} \delta_{i_2 j_1} \delta_{j_2 j_3}$$
$$- \delta_{i_1 j_2} \delta_{j_1 j_3} \delta_{i_2 i_3} + \delta_{i_1 j_3} \delta_{j_1 j_2} \delta_{i_2 i_3} + \delta_{i_1 i_2} \delta_{j_1 j_3} \delta_{i_3 j_2} - \delta_{i_1 j_3} \delta_{i_2 j_1} \delta_{i_3 j_2}], \tag{8}$$

$$I_c = \zeta_1^{i_1} \sum_{L=1}^{5} k_L^{i_1} \frac{\ln(x_1 \cdots x_L)}{(-\ln w)} K_4(2,3,4,5) + \zeta_2^{i_2} \sum_{L=2}^{1} K_L^{i_2} \frac{\ln(x_2 \cdots x_L)}{(-\ln w)} K_4(3,4,5,1) + \text{cyclic perms.}, \tag{9}$$

where, e.g.,

$$K_4(2,3,4,5) = \zeta_2^{i_2} \zeta_3^{i_3} \zeta_4^{i_4} \zeta_5^{i_5} k_2^{j_2} k_3^{j_3} k_4^{j_4} k_5^{j_5} \operatorname{Tr}(R_0^{i_2 j_2} R_0^{i_3 j_3} R_0^{i_4 j_4} R_0^{i_5 j_5}), \tag{10}$$

$$I_d = \left\{ K_4(2,3,4,5) \zeta_1^{i_1} \sum_{l=1}^{\infty} \frac{1}{1-w^l} \left\{ k_2^{i_1} \left[x_2^l - \left(\frac{w}{x_2} \right)^l \right] + k_3^{i_1} \left[(x_2 x_3)^l - \left(\frac{w}{x_2 x_3} \right)^l \right] \right. \right.$$
$$\left. \left. + k_4^{i_1} \left[\left(\frac{w}{x_5 x_1} \right)^l - (x_5 x_1)^l \right] + k_5^{i_1} \left[\left(\frac{\omega}{x_1} \right)^l - x_1^l \right] \right\} + \text{cyclic perms.} \right\} \tag{11}$$

In Eq. (5), g is the coupling constant, Λ^a are the $\frac{1}{2}N(N-1)$ generators of $O(N)$ written in the adjoint representation, and $w = x_1 x_2 x_3 x_4 x_5$. If we change to the disk variables

$$\nu_I = \ln(x_1 \cdots x_I)/\ln w, \quad 1 \leqslant I \leqslant (M-1), \tag{12}$$

$$q = \exp(2\pi^2/\ln w), \tag{13}$$

the Jacobian gives the following in Eq. (5):

$$\left(\prod_{I=1}^{M} dx_I \right) w^{-1} \left(-\frac{2\pi}{\ln w} \right)^5 = 16\pi^3 \frac{dq}{q} \left(-\frac{2\pi^2}{\ln q} \right)^{M-4} \prod_{I=1}^{M-1} \theta(\nu_{I+1} - \nu_I) d\nu_I. \tag{14}$$

The integrand is meromorphic in q since the pole corresponds physically in dilaton emission.[4,5] Hence, noting that

$$\psi_{IJ}=\psi(\nu_J-\nu_I,q), \tag{15}$$

$$\psi(\nu,q)=-\left(\frac{2\pi}{\ln q}\right)\sin\pi\nu\prod_{n=1}^{\infty}\frac{1-2q^{2n}\cos 2\pi\nu+q^{4n}}{(1-q^{2n})^2}, \tag{16}$$

and using $\sum_{I<J}k_I k_J=0$, we see that for the pentagon the leading divergence of the annulus arises from those parts of $I_a+I_b+I_c+I_d$ which generate a single power $(\ln q)$. This arises only from the mode sums of I_b and I_d because if we define the planar mode sum

$$f_P(\nu,\omega)=\sum_{l=1}^{\infty}\frac{1}{1-w^l}[c^l-(w/c)^l], \tag{17}$$

where $c=w^\nu$, then, near the end point $q=0$,

$$f_P(\nu,\omega)=-\frac{(\ln q)}{2\pi}\cot\pi\nu\left[1+O\left(\frac{1}{\ln q}\right)\right], \tag{18}$$

so that each mode sum in Eq. (7) for I_b and Eq. (11) for I_d contributes to the infinity of the annulus.

Note that the nonleading term behaving as $(q\ln q)^{-1}$ as $q\to 0$ would, if present, also diverge. From the physical picture of soft dilaton emission we know that such a unitarity-nonconserving piece must be zero; the algebra underlying this involves I_a and I_c also and will be discussed in Ref. 6.

The infinity of the annulus arises from insertion of Eq. (18) in Eq. (5), and we must now consider its comparison to the pentagonal Moebius-strip diagram. Fortunately, the infinity cancellation takes place term by term in our expressions. A_{5N} is obtained from A_{5P} in Eq. (5) by making the following changes:

(i) Replace the group factor N by (-1).

(ii) Replace ψ_{IJ} by $\psi_{N,IJ}=\psi_N(\nu_J-\nu_I,q)$ with

$$\psi_N(\nu,q)=-\left(\frac{4\pi}{\ln q}\right)\sin(\tfrac{1}{2}\pi\nu)\prod_{n=1}^{\infty}\frac{1-2(-\sqrt{q})^n\cos\pi\nu+q^n}{[1-(-\sqrt{q})^n]^2}. \tag{19}$$

(iii) Replace x_1 by $-x_1$ (we twist the propagator between k_5 and k_1).

(iv) Integrate $d\nu_I$ from 0 to 2, instead of 0 to 1. To combine the infinities, we make the variable changes $\lambda=q^2$ in A_{5P}, and $\lambda=\sqrt{q}$, $\nu_I'=\nu_I/2$ in A_{5N}.

There is, however, also the change that the mode sum in Eq. (17) is replaced by the nonorientable version

$$f_N(\nu,\omega)=\sum_{l=1}^{\infty}\frac{1}{1-(-w)^l}[c^l-(-w/c)^l] \tag{20}$$

$$=-\frac{1}{2}\frac{\ln q}{2\pi}\cot\left(\frac{\pi\nu}{2}\right)\left[1+O\left(\frac{1}{\ln q}\right)\right]. \tag{21}$$

The factor $\frac{1}{2}$ sitting in front of Eq. (21) is the superstring miracle which allows the finiteness to persist for $M>4$. Note that the sign change $x_1\to -x_1$ can always be avoided in $f_N(\nu,\omega)$: If c contains x_1 simply use $c'=w/c$ which does not. Defining $F_5(q^2)$ in the obvious way by substitution of Eq. (18) in Eqs. (7) and (11) and these, with Eq. (14), in Eq. (5) gives the divergent part

$$A_{5P}=N\int_0^1\frac{dq}{q}F_5(q^2)+\text{finite}, \tag{22}$$

while the changes to A_{5N} give

$$A_{5N}=-\tfrac{1}{2}\times 16\int_0^1\frac{dq}{q}F_5(-\sqrt{q})+\text{finite}, \tag{23}$$

where the overall $\frac{1}{2}$ is from Eq. (21) and the 16 arises from the four changes $\nu_I'=\nu_I/2$. Clearly now, the PP prescription leads to finiteness for O(32) just as for $M=4$.

For general numbers of external legs M (and suitably constrained kinematics) there are $M-3$ pieces which contribute to the divergent term for $4\leqslant M\leqslant 7$ and five pieces for all $M\geqslant 8$. For example, $M=6$ has the three pieces

$$R_0^4P(n\neq 0)^2,\quad R_0^3P(S_nS_0)^2,$$
$$R_0^2(S_nS_0)^4,$$

all containing two mode sums. For general $M\geqslant 8$ the five pieces are

$$R_0^4P^{M-4},\quad R_0^3M^{M-5}(S_nS_0)^2,$$
$$R_0^2P^{M-6}(S_nS_0)^4,\quad R_0P^{M-7}(S_nS_0)^6,$$
$$P^{M-8}(S_nS_0)^8,$$

all containing $(M-4)$ mode sums. Thus

$$A_{MP}+A_{MN}=\int_0^1\frac{dq}{q}[NF_M(q^2)-(\tfrac{1}{2})^{M-4}2^{M-1}F_M(-\sqrt{q})]+\text{finite}, \tag{24}$$

giving a finite result for O(32) independent of M.

Concerning our restriction to $k^+=\zeta^+=0$ kinematics, the general configurations with $k^+\neq 0$ have been dealt with in light-cone gauge only for the "simple" case of $M=4$ tree amplitudes.[7] It is expected that, though our calculational technique is valid only for all external momenta and polarization vectors in the transverse space, Eq.

(24) is valid for the most general kinematical configurations.

Although we have carried out our calculations only for external massless gauge bosons, the finite result for diagrams with external massless gauge fermions is to be expected because of supersymmetry. We have presented only an outline of our calculation; more complete details of one-loop amplitudes in this and other superstring theories will appear elsewhere.[6] The continued health of the O(32) model may justify its phenomenological reexamination (e.g., Mani *et al.*[8]).

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Note added.—We have been informed by Professor L. Clavelli (private communication) that he has independently checked the finiteness at one loop of the same O(32) theory, using a covariant formulation.

After this paper was submitted, we realized that the constraint $\zeta^+ = 0$ used in Eq. (3) of the text can be relaxed, allowing the generalization of our results for both parity-nonconserving and parity-conserving amplitudes to all numbers $M \leq 10$ of external ground-state particles.[6]

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