Baryons as Solitons in the Effective Lagrangian of Spontaneously Broken Chiral Symmetry

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I present the calculation of the effective Lagrangian from the σ-quark model on the basis of some reasonable assumptions of the spontaneously broken chiral symmetry, the large-N_c limit, and the nature of the effective-action expansion. In addition to the Skyrme Lagrangian with a unique coupling constant $e=\pi[2(3/N_c)]^{1/2}$ and the Wess-Zumino term, it contains an extra new term $(N_c/96\pi^2)\mathrm{Tr}(\partial_\mu L^\mu)^2$ which must be treated as a perturbative correction. This new term is responsible for the correct pion form factor corresponding to $m_\rho^2=8\pi^2 f_\pi^2(3/N_c)$. This model can be used to describe the rich low-energy dynamics of baryons and mesons with the unique mass scale f_π.

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If quantum chromodynamics (QCD) is the correct theory for the low-energy dynamics of the light hadrons, it must follow the scenario that $SU(3)_L \otimes SU(3)_R$ chiral symmetry must be broken to $SU(3)_V$ spontaneously. The resulting Goldstone bosons which are identified with the nonet of the pseudoscalar mesons must transform as the $(\mathbf{3}^*,\mathbf{3}) \oplus (\mathbf{3},\mathbf{3}^*)$ representation. It has been a challenge for the theorists to construct an effective low-energy theory of hadrons that realizes these symmetry properties and reflects the underlying composite structure of confining quarks and/or antiquark states.

The necessary criterion for a composite particle to be described by an effective Lagrangian at low energy is that its Compton wavelength $1/m$ must be much larger than its size.[1] The only hadrons that qualify naturally are the nonet of pseudoscalar mesons. Therefore, the only sensible effective Lagrangian for the light hadrons is the nonlinear SU(3) σ model,[2,3]

$$\int d^4x\, \mathscr{L}_{\mathrm{eff}} = \int d^4x \left[\tfrac{1}{4} f_\pi^2\, \mathrm{Tr}(\partial_\mu U^\dagger \partial^\mu U) + (\text{four-derivative terms}) + \ldots\right], \qquad (1)$$

where $f_\pi = 94$ MeV is the pion decay constant and $U=\exp(i\lambda_a \Phi_a/f_\pi)$ is a unitary SU(3) matrix that transforms like $(\mathbf{3},\mathbf{3}^*)$ under the chiral $SU(3)_L \otimes SU(3)_R$.

Since it may not be practical to derive this effective Lagrangian from QCD, I shall assume the next level of the effective low-energy dynamics to be the σ-quark model. In this paper I shall present the effective-action expansion of this model and show that, to leading order in the $1/N_c$ expansion (N_c is the number of colors), the four-derivative terms can be determined completely with no arbitrary parameter and that they contain the Wess-Zumino anomaly term,[4] the Skyrme term[5] with renormalization $e=\pi[2(3/N_c)]^{1/2}$ comparable to that determined empirically by Adkins, Nappi, and Witten,[6] and an extra new term which is shown to be responsible for the correct description of the vector form factor in the meson sector and which must be treated as the perturbation correction to the Skyrme model. I further propose that the inclusion of this four-derivative term in the effective-Lagrangian expansion (1) is sufficient to describe low-energy hadron phenomenology with baryons as solitons.[5–8]

The first term in Eq. (1) containing two derivatives is the minimal nonlinear σ model which is completely determined by the underlying symmetry properties of the model. It reproduces the successful current-algebra results in the zero-energy limit. However, this term cannot give the proper structures to hadrons; for example, it can be seen that the pion does not acquire any form factor, and that in the absence of the Wess-Zumino term, processes such as $K\bar{K}\to 3\pi$ and $\pi \to 2\gamma$ are forbidden. The deeper structures of hadrons must come from the dynamics of a more fundamental theory and are manifested as the multiple-derivative terms in the effective expansion Eq. (1). The more derivatives the successive terms contain, the more detailed are the structures of the fundamental dynamics required to calculate these terms. In the large-N_c limit, the four-derivative terms are determined by the representation of the constituent particles, i.e., the quarks, and are independent of the detailed dynamics of their interaction. The Lagrangian for the nonlinear σ-quark model is[9]

$$\int \mathscr{L}\, d^4x = \int d^4x \left\{\tfrac{1}{4} f_\pi^2\, \mathrm{Tr}\partial^\mu U^\dagger \partial_\mu U - \bar{q}[\gamma i^{-1}\partial + m u(x)]q\right\}, \qquad (2)$$

where

$$u(x) = \tfrac{1}{2}(1+\gamma_5)U + \tfrac{1}{2}(1-\gamma_5)U^\dagger = \exp(i\lambda_a \Phi_a \gamma_5/f_\pi)$$

is also unitary and the summation of the color quantum numbers is understood for the quark fields. The N_c dependence can be displayed explicitly by the scaling $f_\pi \to f_\pi (N_c/3)^{1/2}$. The chiral perturbation expansion in powers of $1/f_\pi^2$, the $1/N_c$ expansion, the semiclassical expansion in powers of $\hbar$, and the loop expansion are all equivalent, except for an additional factor of N_c associated with the summation of color in each quark loop. Therefore, in the large-N_c limit the leading contributions are the tree and one-quark-loop graphs. The meson loops and other graphs are suppressed by at least $1/N_c$ and will not be considered in this calculation. The $1/N_c$ expansion of the σ-quark model is therefore completely consistent with the $1/N_c$ expansion of QCD.[10]

It is nontrivial to calculate the effective-action expansion of Eq. (2). Unlike the effective potential, there has not been any simple method to calculate the multiderivative terms in the effective-action expansion.[11] For this reason, there has been an incorrect claim of having derived the Skyrme Lagrangian from the effective action of Eq. (1), and consequently, the determination of the coefficient of the Skyrme term is erroneous.[12] Motivated by this problem, I have formulated a systematic method by which to evaluate the multiderivative terms in the effective-action expansion beyond the effective potential, relying only on the familiar momentum-space and Feynman propagators.[11] I shall refer the reader to Ref. 11 for the details and summarize the essence of this method as applied to this problem.

It is convenient to perform the calculation in the Wick-rotated Euclidean space with $x_0 \to -ix_4$, $\gamma_0 = -i\gamma_4$, and the Euclidean metric $(1,1,1,1)$. The quark fields can be eliminated by explicitly carrying out the functional integration in

$$\int d[q]D[\bar{q}]\exp\left\{-\int d^4x\, \bar{q}(x)[-\gamma i^{-1}\partial + mu(x)]q(x)\right\} = \exp(-N_c \operatorname{Tr}\ln S_F^{-1}),$$

where $S_F(x,y)$ is the Euclidean Green's function defined by

$$[-\gamma i^{-1}\partial + mu(x)]S_F(x,y) = \delta^4(x-y).$$

The crucial step for the effective-action expansion is the formal solution for the Green's function $S_F(x,y)$ as a function of the background field $u(x)$. As shown in Ref. 11, the solution for the Green's function in the momentum space is given as if the background field u is constant, except that the argument of u is replaced by $x + i\,\partial/\partial p$:

$$S_F(x,y) = \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot y}\left[-\gamma\frac{1}{i}\partial + mu(x)\right]^{-1} e^{-ip\cdot x} = \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot(y-x)} S(p,x),$$

where

$$S(p,x) = [\gamma\cdot p + mu^\dagger(x + i\,\partial/\partial p)]\left[p^2 + m^2 - m\gamma\frac{1}{i}\partial u^\dagger(x + i\,\partial/\partial p)\right]^{-1}$$

$$= \frac{\gamma\cdot p + mu^\dagger(x)}{p^2+m^2}\left\{\sum_{l=0}^{\infty}\left[m\sum_{n=0}^{\infty}\frac{1}{n!}\left(\gamma\frac{1}{i}\partial\partial_{i_1}\cdots\partial_{i_n}u^\dagger(x)\right)i\frac{\partial}{\partial p_{i_1}}\cdots i\frac{\partial}{\partial p_{i_n}}\frac{1}{p^2+m^2}\right]^l\right\}. \tag{3}$$

This expansion can be applied directly to evaluate the variation of the effective action,

$$\delta \operatorname{Tr}\ln S_F^{-1} = \int d^4x\, m \operatorname{Tr}\delta u(x) S_F(x,x) = \int d^4x\, m\operatorname{Tr}\delta u(x)\int\frac{d^4p}{(2\pi)^4}S(p,x).$$

Apart from the tensor structure, which can be treated easily by the standard symmetry considerations, the remaining momentum integrations are of the same general type as given in Ref. 11. Keeping only terms up to four derivatives and taking the trace of the γ matrices I obtain the quark-loop contribution to the effective Lagrangian,

$$\delta\mathscr{L}_q = \frac{N_c}{16\pi^2}\left[\frac{1}{\epsilon}+\psi(1)\right]m^2\operatorname{Tr}(\partial^2U^\dagger\delta U + \delta U^\dagger\partial^2U) + \frac{N_c}{96\pi^2}\operatorname{Tr}(\partial^2U^\dagger\delta U + \delta U^\dagger\partial^4U)$$
$$+\frac{N_c}{48\pi^2}\operatorname{Tr}(2U_l^\dagger U_{lk}U_k^\dagger\delta U + 2U_{ll}^\dagger U_kU_k^\dagger\delta U - U_k^\dagger U_{ll}U_k^\dagger\delta U + 2U_lU_{lk}^\dagger U_k\delta U^\dagger$$
$$+2U_{ll}U_k^\dagger U_k\delta U^\dagger - U_kU_{ll}^\dagger U_k\delta U^\dagger) + \frac{N_c}{48\pi^2}i\epsilon_{ijkl}\operatorname{Tr}(U^\dagger U_iU_j^\dagger U_kU_l^\dagger\delta U), \tag{4}$$

where the subscript is used to denote the derivative, $U_i = \partial_i U$, etc. Since $U^\dagger U = I$, the variations δU and $\delta U^\dagger$ are not independent. This constraint has been used in arriving at the expression Eq. (4). The first term is infinite and can be absorbed into the σ-model Lagrangian by the renormalization of f_π. The last term is the Wess-Zumino anomaly term with coefficient exactly in agreement with Witten's analysis.[7] Because of the unitary constraint it is not possible to perform the functional integration in U for the Wess-Zumino term.[7] However, if instead the calculation is carried out in five-dimensional space, the Wess-Zumino term would be

$$\delta \mathcal{L}_{\text{W-Z}} = (N_c/240\pi^2)\, i\epsilon_{ijklm} \operatorname{Tr} U_i^\dagger U_j U_k^\dagger U_l U_m^\dagger \delta U,$$

which can be integrated exactly to give the action

$$(N_c/240\pi^2)\, i\epsilon_{ijklm} \int_{v(4)} d^5x \operatorname{Tr} U_i^\dagger U_j U_k^\dagger U_l U_m^\dagger U. \tag{5}$$

The integration is over the five-dimensional volume bounded by the four-dimensional space, which, for example, can be the half-space in five-dimensions, with $U(x_5=0) = U$ and $U(x_5=\infty) = 0$. The variation on Eq. (5) is a total divergence and one can convert the integral into a surface integral to recover the result of Eq. (4).

When we add the contribution from the meson part, Eq. (4) can be rewritten as

$$\delta \mathcal{L} = -i \operatorname{Tr}(\partial_i J_{Li} \delta U U^\dagger) = -i \operatorname{Tr}(\partial_i J_{Ri} U^\dagger \delta U).$$

One can apply the Wick rotation back to the Minkowski space and obtain the left-handed effective current in terms of $L_\mu = i(\partial_\mu U) U^\dagger$:

$$J_L^\alpha = \tfrac{1}{2} f_\pi^2 L^\alpha + \frac{N_c}{48\pi^2}\{-2\partial^\mu\partial^\alpha L_\mu + \partial^\mu\partial_\mu L^\alpha - 3i[L^\alpha, \partial^\mu L_\mu] + i[L^\mu, \partial^\alpha L_\mu + \partial_\mu L^\alpha] - i\epsilon^{\alpha\beta\gamma\delta} L_\beta L_\gamma L_\delta\}, \tag{6}$$

while the right-handed effective current J_R^α can be obtained from J_L^α by replacing L by R, with $R_\mu = i(\partial_\mu U^\dagger) U$. The Wess-Zumino term has normalization identical to that obtained by Witten.[7] The baryon current is given by

$$B^\alpha = \operatorname{Tr}[J_R^\alpha + J_L^\alpha] = -i(N_c/24\pi^2)\epsilon^{\alpha\beta\gamma\delta} \operatorname{Tr} L_\beta L_\gamma L_\delta. \tag{7}$$

The effective action can be obtained directly from Eq. (4):

$$S_{\text{eff}} = \int d^4x \operatorname{Tr}\left(\frac{1}{4} f_\pi^2 L_\mu L^\mu + \frac{N_c}{96\pi^2}[3(\partial_\mu L^\mu)^2 - (\partial_\mu L_\nu)(\partial^\mu L^\nu) - (\partial_\mu L_\nu)(\partial^\nu L^\mu)]\right) - \frac{N_c}{240\pi^2}\int_{v(4)} d^5x\, \epsilon_{\mu\nu\alpha\beta\gamma} \operatorname{Tr} L^\mu L^\nu L^\alpha L^\beta L^\gamma. \tag{8}$$

Equation (8) may be rewritten by use of the identity $\partial_\mu L_\nu - \partial_\nu L_\mu = i[L_\nu, L_\mu]$. Then S_{eff} becomes

$$S_{\text{eff}} = \int d^4x \operatorname{Tr}\{\tfrac{1}{4} f_\pi^2 L_\mu L^\mu + (N_c/192\pi^2)\{2(\partial_\mu L^\mu)^2 + [L_\mu, L_\nu]^2\} + \text{W-Z term}. \tag{9}$$

This action differs from the Skyrme model even at the SU(2) level, where the W-Z term vanishes identically. If one simply ignores the additional term of $(\partial_\mu L^\mu)^2$, the coefficient $N_c/192\pi^2$ can be identified with $1/32e^2$ as defined in Ref. 6. Then I obtain $e = \pi[2(3/N_c)]^{1/2} = 4.43$ to be compared to $e = 5.45$ estimated by the masses of N and Δ in Ref. 6.

The new term can be rewritten with the identity

$$\operatorname{Tr}(\partial_\mu L^\mu)^2 = \operatorname{Tr}\partial^2 U^\dagger \partial^2 U - \operatorname{Tr}(L_\mu L^\mu)^2.$$

The static energy is given by

$$E = \tfrac{1}{4} f_\pi^2 \operatorname{Tr}\int d^3x \{\nabla U \cdot \nabla U^\dagger + M^{-2}[-\nabla^2 U \nabla^2 U^\dagger + \tfrac{1}{2}(\partial_i L_j - \partial_j L_i)^2 + (\mathbf{L}\cdot\mathbf{L})^2]\}, \tag{10}$$

where the characteristic mass scale M is defined by

$$M = 2\pi f_\pi [2(3/N_c)]^{1/2} = 2f_\pi e = 840 \text{ MeV}.$$

The term with the negative sign renders the expression not positive definite, which may jeoparidize the stability of the soliton solutions. In order to understand the nature of this negative sign, it is necessary to examine its effects on the meson sector. From the currents given by Eq. (5), the meson vector form factor is $F(q^2) = 1 + q^2/M^2$. The coefficient of the term with the negative sign in the energy is in fact the slope of the vector form factor! *Therefore, independent of any particular model, if the vector form factor of the pion has the correct sign for its slope, the energy* *cannot be positive definite.* In this model not only is the sign of the slope correct, but the magnitude is close to that of the vector-meson–dominance model,[9] $M \approx m_\rho$.

The apparent inconsistency may be understood in the following way. Both the energy and the form factor are approximations in the low-energy expansion. The form factor cannot be correct for large q^2. In the spacelike region, $F(q^2)$ is not positive definite. However, if one uses Padé's approximation, the form factor becomes[9] $F(q^2) = (1 - q^2/M^2)^{-1}$. This is what one would expect for a vector-dominance form factor with $M \approx m_\rho$. More importantly, the form factor becomes positive definite in the spacelike region. The indefiniteness is, hence, an artifact of the low-energy expansion.

The same analysis applied to (9) yields a positive definite energy,

$$E = \tfrac{1}{4} f_\pi^2 \operatorname{Tr} \int d^3x \{ \nabla U \cdot (1 - \nabla^2/M^2)^{-1} \nabla U^\dagger + M^{-2} [\tfrac{1}{2} (\partial_i L_j - \partial_j L_i^2) + (\mathbf{L} \cdot \mathbf{L})^2] \}. \tag{11}$$

Furthermore, a simple application of the scaling argument[13] shows that Eq. (11) can have stable, finite-energy soliton solutions.

As in many other physical systems, the instability of the ground state may be caused by truncation of the low-energy expansion. A well-known example occurs in the nonrelativistic reduction of either the Dirac equation or the Bethe-Salpeter equation. The $-P^4/8m^3$ term or the spin-orbit term from the Coulomb potential can only be treated consistently as a perturbation term. Similarly, in this QCD model, the instability induced by the new term should not prevent the use of this low-energy expansion, Eq. (9), as a vital approximation for the low-energy hadron phenomenologies, as long as it can be treated consistently as a perturbation term to the Skyrme model.

If one can accept this viewpoint, one can begin to appreciate the elegance of Eq. (9) and to entertain the real possibility that the richness of low-energy hadron physics can be summarized by a simple model with a unique mass scale f_π. The phenomenological analysis of this model will be reported in future publications.

After this work was completed, I learned that Aitchison and Fraser[14] and independently MacKenzie, Wilczek, and Zee[15] have also computed the effective-action expansion for the SU(2) σ model from the quark-loop contribution. Our methods of calculation and interpretations are completely different. I would like to thank A. Zee for an advance copy of the manuscript of their work. This work was supported in part by the U.S. Department of Energy.

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