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Many-Particle Translational Symmetries of Two-Dimensional Electrons at Rational Landau-Level Filling

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In contrast to previous treatments, a new analysis of two-dimensional many-electron systems subject to periodic boundary conditions in a magnetic field leads to a fully two-dimensional structure of the quantum numbers at rational Landau-level filling. The structure of the new symmetry analysis has an intrinsically many-particle character. Full agreement between numerical studies of quantized-Hall-effect systems in periodic and spherical geometries is achieved, and the problem of ground-state degeneracy is clarified.

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The quantized-Hall-effect (QHE) phenomenon[1] has focused theorists' attention on two-dimensional (2D) many-electron systems in a magnetic field. It was long ago shown[2] that the structure of the translation-symmetry group of a charged particle in a magnetic field is richer than that in the zero-field case, and if a substrate potential is present, has simple features only if the potential is periodic with a rational flux per unit cell. In this Letter, I report that the translational symmetries of the *many*-particle system with *rational flux (q/p) per particle* (in units of the flux quantum $\Phi_0 = h/e$, so that $\nu = p/q$ is the Landau-level filling factor) also have a previously unsuspected richer structure—intrinsically many particle in character—than that of the one-particle system. This structure emerges from an analysis of the application of periodic boundary conditions to a finite system, and develops fully as the thermodynamic limit is taken at fixed ν. The intrinsically many-particle character of the translational-symmetry quantum numbers appears to be novel, and is formally a consequence of the fact that in the presence of a magnetic field, the translation operators define a projective (ray) representation[3] of the translation group with a number-dependent factor system.

The full characterization of symmetry is essential for the understanding of physical systems, and the new analysis resolves puzzling contradictions between previous studies of 2D many-electron systems which have appeared to depend significantly on the choice of gauge and boundary conditions. Studies using explicitly isotropic formalisms (disk[4] and sphere[5] geometries) have provided a rather successful picture of the QHE, but are difficult to reconcile fully with other studies[6–8] using a periodic boundary condition (PBC). The former studies describe an essentially nondegenerate ground state, while the q-fold ground-state degeneracy reported in the latter studies has been taken by some authors[7,9] as a feature intrinsic to the fractional QHE. There is also little apparent resemblence between excitation spectra reported in spherical[5] and periodic[8] geometries. The reason for these discrepancies turns out to be the fact that previous authors using the conventional PBC formalism have simply applied the one-particle symmetry analysis,[2] overlooking the new many-particle features reported here. I find that (at *rational ν only*) states may be characterized by a 2D wave vector **k**, while earlier work has only recognized the component of **k** along the direction singled out by the Landau gauge as a quantum number. The q-fold ground-state degeneracy emphasized by some authors is identified as a center-of-mass degeneracy common to all states, unrelated to whether or not the system has an incompressible QHE ground state.

While eigenfunctions and eigenvalues obtained in previous numerical studies of finite systems with PBC's remain correct, the earlier classification of quantum numbers is essentially meaningless. In addition, use of the full symmetry allows such studies to be repeated and extended with much smaller computing effort, and I illustrate this by presenting the excitation spectrum of the six-particle $\nu = \frac{1}{3}$ system, now remarkably similar to results obtained on the sphere.[5] *Direct* comparison of the ground state with the recent periodic reformulation[10] of the Laughlin-Jastrow wave function[4] is made for the first time, and incompressible QHE states are characterized as $\mathbf{k}=0$ states—the only value of $\mathbf{k}$ at which point-rotational degeneracy is absent.

In a uniform magnetic field $B_0\hat{\mathbf{z}}$ normal to the 2D system, the operator $t_i(\mathbf{a})$ that translates particle i by $\mathbf{a}$ and commutes with the dynamical momentum $\boldsymbol{\pi}_i = -i\hbar\nabla_i - e\mathbf{A}(\mathbf{r}_i)$ obeys the algebra[2]

$$t_i(\mathbf{a}+\mathbf{b}) = t_i(\mathbf{b})t_i(\mathbf{a})\exp(i\hat{\mathbf{z}}\cdot\mathbf{a}\times\mathbf{b}/2l^2),$$

where l is the "magnetic length" $(\hbar/eB_0)^{1/2}$; $t(\mathbf{a}) = \exp(i\mathbf{a}\cdot\mathbf{K}_i/\hbar)$, where $\mathbf{K}_i = \boldsymbol{\pi}_i - \hbar\hat{\mathbf{z}}\times\mathbf{r}_i/l^2$ is the "pseudomomentum." Periodic boundary conditions require *physical quantities* to be invariant under translation of any particle coordinate $\mathbf{r}_i$ by $\mathbf{L}_{mn}$, where the set $\{\mathbf{L}_{mn}\} = \{m\mathbf{L}_1 + n\mathbf{L}_2;\ m,n \text{ integer}\}$ defines a 2D Bravais lattice with primitive unit cell area $\hat{\mathbf{z}}\cdot\mathbf{L}_1\times\mathbf{L}_2 = 2\pi N_s l^2$, traversed by N_s quanta of magnetic flux. It will be convenient to define *primitive* translations $\mathbf{L}_{mn}$ as those where $\lambda\mathbf{L}_{mn}$, $0<\lambda<1$, is not a lattice vector (i.e., those where m and n have no common divisor greater than unity, and are not both zero).

All physical quantities, including the Hamiltonian, can be expressed in terms of the gauge-invariant products $\langle\{\mathbf{r}_i\}|\alpha\rangle\langle\beta|\{\mathbf{r}_i\}\rangle$, where $\psi_\alpha(\{\mathbf{r}_i\}) = \langle\{\mathbf{r}_i\}|\alpha\rangle$ are the Schrödinger wave functions. The PBC becomes the condition $[|\alpha\rangle\langle\beta|, t_i(\mathbf{L}_{mn})] = 0$, which is a selection rule that $|\alpha\rangle\langle\beta|$ is constructed from states that are both simultaneously eigenfunctions [with the *same* eigenvalues $\exp(i\theta^i_{mn})$] of all $t_i(\mathbf{L}_{mn})$. The additional selection rule that states are symmetric or antisymmetric under exchange of identical particles i,j requires that $\exp(i\theta^i_{mn}) = \exp(i\theta^j_{mn})$. Simultaneous diagonalization of the $t_i(\mathbf{L}_{mn})$ requires that N_s be integral.

Physical states of N_e identical particles with periodic boundary conditions thus belong to one of a two-parameter family of equivalent Hilbert spaces $\mathscr{H}(\theta_1,\theta_2)$ where $t_i(\mathbf{L}_{mn}) = \exp(i\theta_{mn})$, $\theta_{mn} = \pi mnN_s + m\theta_1 + n\theta_2$. The filling factor ν is N_e/N_s, where $N_e = pN$ and $N_s = qN$ have a maximum common divisor N.

The center-of-mass (c.m.) translation operator is the product $T(\mathbf{a}) = \prod_i t_i(\mathbf{a})$. Only the set $\{T(\mathbf{L}_{mn}/N_s)\}$ acts within a given Hilbert space $\mathscr{H}(\theta_1,\theta_2)$; other c.m. translations map states from one space to another. The PBC translation operators can be factorized: $t_i(\mathbf{L}_{mn}) = T(q\mathbf{L}_{mn}/pN_s)\tilde{t}_i(\mathbf{L}_{mn})$, where

$$\tilde{t}_i(\mathbf{a}) = \prod_j t_i(\mathbf{a}/N_e)t_j(-\mathbf{a}/N_e), \quad \prod_i \tilde{t}_i(\mathbf{a}) = 1.$$

The operators $\tilde{t}_i(\mathbf{a})$ are *invariant under center-of-mass translations,* and will provide the fundamental quantum numbers of the system. The largest set of these operators which act within a physical Hilbert space is $\{\tilde{t}_i(p\mathbf{L}_{mn})\}$, which can be simultaneously diagonalized. I define the reciprocal vector $\mathbf{k}$, so that for the set of *primitive* translations $\mathbf{L}_{mn}$, the eigenvalues are

$$\tilde{t}_i(p\mathbf{L}_{mn}) = (-1)^{pq(N_e-1)}\exp(-iq\mathbf{k}\cdot\mathbf{L}_{mn}/N_s).$$

The prefactor (derived below) fixes the $\mathbf{k}=0$ states; the scale of $\mathbf{k}$ is chosen to be standard so that the action of $\sum_i \exp(i\mathbf{Q}\cdot\mathbf{r}_i)$ on an eigenfunction of the $\{\tilde{t}_i(p\mathbf{L}_{mn})\}$ increases $\mathbf{k}$ by $\mathbf{Q}$, provided that $\exp(i\mathbf{Q}\cdot\mathbf{L}_{mn}) = 1$, so $\mathbf{Q}$ is allowed by the PBC.

Allowed $\mathbf{k}$ values for identical-particle systems satisfy $\exp(ip\mathbf{k}\cdot\mathbf{L}_{mn}) = 1$; there are N^2 distinct values of $\mathbf{k}$ for a finite system, forming a mesh of area $2\pi/Nql^2$ in reciprocal space, in a "Brillouin zone" of area $2\pi N/ql^2$. In the thermodynamic limit $N\to\infty$ at fixed p/q, i.e., at *rational Landau-level filling,* a uniform cover of *all* reciprocal space is obtained. Note that in the case of *irrational* filling factors, obtained as the limit of a sequence where N_e and N_s have only the common divisor $N=1$, this 2D reciprocal space cannot be constructed, and the earlier symmetry analysis[6] is complete.

I now specialize to translationally invariant systems with PBC's, so that the Hamiltonian commutes with all $T(\mathbf{a})$, and the set $\{t_i(\mathbf{L}_{mn})\}$. [This last requirement can be satisfied by Fourier transformation of the pair interaction, and reconstruction of it by discretization of the reverse Fourier integral over reciprocal space to a sum over the mesh of $\mathbf{Q}$ values where $\exp(i\mathbf{Q}\cdot\mathbf{L}_{mn}) = 1$, but this is not a unique prescription.] Because of translational invariance, the spectrum of the Hamiltonian is independent of the parameters θ_1, θ_2, and $\mathbf{k}$ is a good quantum number. The total kinetic energy $\sum_i|\hat{\mathbf{z}}\times\boldsymbol{\pi}_i|^2/2m$ can be separated into a center-of-mass term $H^{\text{c.m.}} = |\sum_i\hat{\mathbf{z}}\times\boldsymbol{\pi}_i|^2/2mN_e$, plus a relative-motion term which combines with the interaction term to give

$$H^{\text{rel}} = \sum_{i<j}\left\{\frac{1}{2mN_e}|\hat{\mathbf{z}}\times(\boldsymbol{\pi}_i-\boldsymbol{\pi}_j)|^2 + \frac{1}{N_s l^2}\left[\sum_{\mathbf{Q}}\tilde{V}(Q)e^{i\mathbf{Q}\cdot(\mathbf{r}_i-\mathbf{r}_j)}\right]\right\}.$$

The relative-motion variables of H^{rel} are independent of the center-of-mass variables of $H^{\text{c.m.}}$: The many-particle wave function can thus be factorized into a product $\Psi^{\text{c.m.}}\otimes\Psi^{\text{rel}}$. The principal symmetry quantum number of Ψ^{rel} is $\mathbf{k}$, and in general[11] no states of H^{rel} with the same $\mathbf{k}$ are degenerate; the energy levels of $H^{\text{c.m.}}$ are classified by the c.m. Landau-level index, but in general have additional degeneracy associated with action of the Hilbert-space–preserving c.m. translations $T(\mathbf{L}_{mn}/N_s)$. Once θ_1, θ_2, and $\mathbf{k}$ have been fixed, this residual c.m. degeneracy is q-fold: $\Psi^{\text{c.m.}}$ is then determined to be a simultaneous eigenfunction of the set $\{T(qr\mathbf{L}_{mn}/N_s)\}$, where $\mathbf{L}_{mn}$ is primitive, and r is any integer, with eigenvalues

$$(-1)^{pqr(N_e-1)}\exp(ipr\theta_{mn}+iqrN_s^{-1}\mathbf{k}\cdot\mathbf{L}_{mn}).$$

The residual c.m. degeneracy can be resolved by requiring $\Psi^{\text{c.m.}}$ to be an eigenstate of $T(\mathbf{L}^0/N_s)$, where $\mathbf{L}^0$ is some particular primitive translation. Since the qth power of this eigenvalue is already determined, it takes q possible values. The combined set $\{T((q\mathbf{L}_{mn}+r\mathbf{L}_0)/N_s)\}$ is a maximally commuting set, and $\Psi^{\text{c.m.}}$ cannot be an eigenfunction of any further $T(\mathbf{L}_{mn}/N_s)$. Various other resolutions of the q-fold c.m. degeneracy are possible.

It should be emphasized that this q-fold c.m. degeneracy is a purely group-theoretical consequence of the imposition of PBC's on a translationally invariant system, and *quite without physical significance.* It is a degeneracy common to *every* eigenvalue of the Hamiltonian belonging to a given subspace $\mathscr{H}(\theta_1,\theta_2)$, and related to the degeneracy between subspaces. It is present *independent of the physical nature of the ground state of* H^{rel}, whether it is of the fluid type and exhibits the QHE, or a solid type and does not.

Imposition of PBC's breaks isotropy of the system, which is only recovered as $N\to\infty$. However, the finite system still has some point symmetry. The following statements can be made: (1) The spectrum as a function of $\mathbf{k}$ has the full point symmetry of the PBC Bravais lattice; however reflections reverse the magnetic field, and are only symmetries of H^{rel} when coupled with the antiunitary time-reversal operation. (2) The only quantum numbers of H^{rel} that can be specified in addition to $\mathbf{k}$ are those associated with twofold, threefold, fourfold, or sixfold rotations at $\mathbf{k}=0$ and certain other high symmetry points in the "Brillouin zone" of a finite system. (3) Unless Landau-level particle-hole symmetry is present,[11] conjugate representations of these point-rotation groups are not degenerate, and the only degeneracy of states of H^{rel} is that between states with different $\mathbf{k}$ vectors related by point symmetries of the Bravais lattice.

These considerations fix the definition of the $\mathbf{k}=0$ state. In the thermodynamic limit, this is the only value of $\mathbf{k}$ not related to any other by point symmetry. In the case of the highest-symmetry PBC—the hexagonal Bravais lattice—there is only one such point in $\mathbf{k}$ space, the one at which sixfold rotation symmetry is present. This has been taken as the definition of the $\mathbf{k}=0$ point, and can also be shown to remain consistently nondegenerate as the Bravais lattice is adiabatically varied. The hexagonal lattice may be specified by choosing $|\mathbf{L}_1|=|\mathbf{L}_2|=2\mathbf{L}_1\cdot\mathbf{L}_2/|\mathbf{L}_1|$. Under a $\pi/3$ rotation, $\mathbf{L}_1\to\mathbf{L}_2$, $\mathbf{L}_2\to(\mathbf{L}_2-\mathbf{L}_1)$. The invariant point is the solution of

$$\tilde{t}_i(p\mathbf{L}_1)=\tilde{t}_i(-p\mathbf{L}_1)=\tilde{t}_i(p\mathbf{L}_2)=\tilde{t}_i(p\mathbf{L}_2-p\mathbf{L}_1),$$

which can be reexpressed as $(-1)^{pq(N_e-1)}\tilde{t}_i(p\mathbf{L}_2)\tilde{t}_i\times(-p\mathbf{L}_1)$. Thus when $\mathbf{k}=0$, $\tilde{t}(p\mathbf{L}_1)=\tilde{t}(p\mathbf{L}_2)=(-1)^{pq(N_e-1)}$, and *all* $\tilde{t}(p\mathbf{L}_{mn})$ with $\mathbf{L}_{mn}$ primitive are found to have the same value.

QHE incompressible-fluid states can now be characterized as states where H^{rel} has a *nondegenerate* $\mathbf{k}=0$ ground state with a finite gap for all excitations. The interpretation of $\mathbf{k}$ is particularly clear in the case of a filled Landau level, where $\hbar\mathbf{k}$ is the sum of the pseudomomenta $\mathbf{K}_i$ of particles excited to higher Landau levels, minus those of the empty states (holes) in the filled level. Provided that the numbers of particle and hole excitations are equal, this is invariant under c.m. translations, which add a constant pseudomementum to all single-particle states, empty or full, and has commuting components.

The $\nu=\frac{1}{3}$ QHE state is modeled by the Laughlin-Jastrow (LJ) wave function.[4] The periodic form of this is now available,[10] and becomes the *exact* ground state in the limit $\hbar eB_0/m\to\infty$ for a model "hard-core" interaction[5] corresponding to the pseudopotential $\tilde{V}(Q)=\text{const}-2V_1Q^2l^4$ ($V_1>0$ is the pairing energy of two spin-polarized electrons in the closest pairing state, while all other pairing energies are zero). This state emerges very clearly in numerical studies using the hard-core pseudopotential, as it has exactly zero potential energy, all other states at $\nu=\frac{1}{3}$ having energies proportional to V_1.

To emphasize the practical importance of the new symmetry analysis presented here, I have carried out a numerical diagonalization of the $N_e=6$ system of spin-polarized electrons in the lowest Landau level at $\nu=\frac{1}{3}$. A square PBC Bravais lattice and Coulomb interactions were used. Details, as well as results for larger systems with various ν and PBC geometries, will be described elsewhere. The low-lying excitation spectrum is shown in Fig. 1. Use of a basis of $\mathbf{k}$ eigenstates reduces matrix dimensions by about N: In the present example, the largest matrix had dimension 176, while by use of the methods of Refs. 4–6 dimensions are 1026 and greater.

The ground state of this six-particle system was directly compared to the periodic LJ state: In line with disk[4] and sphere[5] geometry studies, the projection on the LJ state was 98.81%. The collective excitation, with its dispersion that has a characteristic minimum[5,12] at $|\mathbf{k}|\simeq 1.4l^{-1}$, stands out very clearly, and in this new presentation the spectrum is strikingly similar to that obtained in the spherical geometry.[5] Indeed, both ground-state properties and excitation

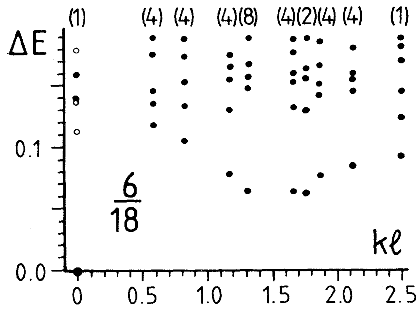

FIG. 1. Low-lying excitation energies (in units $e^2/4\pi\epsilon l$) vs $|\mathbf{k}|$ for the spin-polarized six-electron lowest-Landau-level system at $\nu=\frac{1}{3}$, with Coulomb interactions $V(Q)=e^2/4\pi\epsilon Q$ and square periodic boundary conditions. Pairing degeneracies of $\mathbf{k}$ points are shown at the top. At $\mathbf{k}=0$, the rotational quantum number ΔM(mod 4) is indicated by solid points ($\Delta M=0$) and open points ($\Delta M=2$).

spectra were found to be very insensitive to variation of the PBC geometry away from the square lattice. Rotational quantum numbers ΔM (defined modulo 4 for the square PBC lattice) are indicated for $\mathbf{k}=0$ states. The lowest excited state at $\mathbf{k}=0$ (presumably the long-wavelength limit of the collective excitation dispersion) has $|\Delta M|=2$. It may be remarked that a $\mathbf{k}=0$ excitation *cannot* be described by the Feynman-Bijl *Ansatz* of Girvin, MacDonald, and Platzman,[12] and an interpretation of the $\mathbf{k}\to 0$ limit of the collective mode is still an open problem.

As in Ref. 5, I also examined the effect of changing the short-range pseudopotential component of the interaction by adding a "hard-core" term to the Coulomb interaction. A first-order transition (this time marked by a change of ground-state point-rotation symmetry) to a gapless, compressible state (the Wigner lattice ?) was again observed for $\Delta V_1 \lesssim -0.10e^2/4\pi\epsilon l$.

Finally, I mention results at $\nu=\frac{1}{2}$, where systems of as many as ten particles can now be diagonalized. In contrast to the $\nu=\frac{1}{3}$ case, the ground state is not generally found at $\mathbf{k}=0$, but at general $\mathbf{k}$ points that move and change discontinuously as the PBC geometry is varied. Low-lying excitations are quasidegenerate with the ground states, and no gap structures are apparent. The detailed level structure depends very sensitively on PBC geometry and N, and no clear picture emerges from the finite-system study.

In summary, I have described novel particle-number–dependent features of the translational-symmetry analysis of 2D many-particle systems in a magnetic field, with rational magnetic flux per particle. The new analysis resolves discrepancies between earlier finite-system studies[6–8] using periodic boundary conditions and those using spherical geometry.[5] In particular, the q-fold ground-state degeneracy of finite periodic systems is clearly seen as a group-theoretical consequence of periodic boundary conditions without physical implications. Significant reduction of the computational task of studying finite systems is also obtained, as is a correct classification of the excitation spectrum.

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