Equations of Motion for the Heterotic String Theory from the Conformal Invariance of the Sigma Model

Ashoke Sen[a]
Fermi National Accelerator Laboratory, Batavia, Illinois 60510
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In the presence of arbitrary background gauge, gravitational, and antisymmetric tensor fields, the heterotic string may be described by a two-dimensional local field theory. It is conjectured, and verified to a certain approximation, that the conditions on the background fields for this model to have vanishing β function are identical to the equations of motion for the massless fields. In particular, the appearance of the Chern-Simons three-forms in the classical equations of motion is shown to be related to the chiral anomaly in two-dimensional gauge theories.

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It was shown in a previous paper[1] that in the presence of arbitrary background gauge, gravitational, and antisymmetric tensor fields, the heterotic string[2] may be described by a two-dimensional field theory with $N=\frac{1}{2}$ supersymmetry, given by the action

$$S_0=(1/2\pi)\int d\tau\int_0^\pi d\sigma[g_{ij}(X)\{\partial_\alpha X^i\partial^\alpha X^j+i\bar{\lambda}^i\rho^\alpha(D_\alpha\lambda)^j\}+B_{ij}(X)\epsilon^{\alpha\beta}\partial_\alpha X^i\partial_\beta X^j-iS_{ijk}(X)\bar{\lambda}^i\rho^\alpha\lambda^j\epsilon_{\alpha\beta}\partial^\beta X^k$$
$$+i\delta_{st}\bar{\psi}^s\rho^\alpha\partial_\alpha\psi^t+\bar{\psi}^s(T^M)_{st}\rho^\alpha\psi^t\{A_i^M\partial_\alpha X^i-\tfrac{1}{4}iF_{il}^M\bar{\lambda}^l\rho_\alpha\lambda^i\}], \tag{1}$$

where g_{ij} is the background metric, B_{ij} is the background antisymmetric tensor field, and A_i^M is the background gauge field. The X^i's are the eight bosonic fields, the λ^i's are the eight left-handed Majorana fermions, and the ψ^s's are 32 right-handed Majorana fermions, belonging to the fundamental **(32)** representation of SO(32), or the **(16, 1)** ⊕ **(1, 16)** representation of the subgroup SO(16) ⊗ SO(16) of the group $E_8\otimes E_8$, depending on which particular type of heterotic string is under consideration. The ρ^α's are the two-dimensional Dirac matrices, the T^M's are the generators of the gauge group, and

$$S_{ijk}=\tfrac{1}{2}(B_{ij,k}+B_{jk,i}+B_{ki,j}), \tag{2}$$

$$(D_\alpha\lambda)^j=\partial_\alpha\lambda^j+\Gamma^j{}_{kl}\lambda^k\partial_\alpha X^l=\partial_\alpha\lambda^j+\tfrac{1}{2}g^{ji}(g_{ik,l}+g_{il,k}-g_{kl,i})\lambda^k\partial_\alpha X^l, \tag{3}$$

$$F_{il}^M=A_{i,l}^M-A_{l,i}^M+f^{MNP}A_i^NA_l^P. \tag{4}$$

Although expression (1) of the effective action was derived in the weak-field approximation, I calculated the ultraviolet-divergent part of the effective action in the theory described by Eq. (1) without making any approximation. The result is

$$\left\{i\int\frac{d^2k}{(2\pi)^2}\,\frac{1}{k^2+i\epsilon}\right\}[\tfrac{1}{2}\tilde{R}_{ik}{}^k{}_j(\partial_\alpha X^i\partial^\alpha X^j-\epsilon^{\alpha\beta}\partial_\alpha X^i\partial_\beta X^j)-\tfrac{1}{4}\bar{\psi}\rho^\alpha T^M\psi\{D^kF_{kl}^M-f^{MNP}A^{Nk}F_{kl}^P-S^{ij}{}_lF_{ji}^M\}\partial_\alpha X^l], \tag{5}$$

plus its $N=\frac{1}{2}$ supersymmetric extension. Here D denotes the covariant derivative which includes the spin connection, but not the gauge connection. $\tilde{R}$ is the generalized curvature[3] defined as

$$\tilde{R}_{ijkl}=R_{ijkl}+D_kS_{ijl}-D_lS_{ijk}+S_{mik}S^m{}_{lj}-S_{mil}S^m{}_{kj}, \tag{6}$$

where R is the Riemann tensor.

It was pointed out that the criterion for the vanishing of the one-loop divergence is identical to the equations of motion in the weak-field approximation.[1,4] In this paper, I propose that this correspondence holds beyond the weak-field approximation, and that the classical equations of motion of the full-fledged string theory are identical to the criteria for the vanishing of the β function in the model described by (1). Evidence is produced in support of this conjecture from the explicit one- and two-loop results for the vanishing of the β functions in this model. First, I shall make the following important observations, which will be useful in the analysis.

(i) The action (1) transforms as

$$S\rightarrow S/\Lambda \tag{7}$$

under

$$g_{ij}\rightarrow g_{ij}/\Lambda,\quad B_{ij}\rightarrow B_{ij}/\Lambda,$$
$$\delta_{st}\rightarrow\delta_{st}/\Lambda,\quad (T^M)_{st}\rightarrow(T^M)_{st}/\Lambda, \tag{8}$$

while all other quantities remain unchanged. Thus, in

l-loop order, the effective action must transform as

$$S_{\rm eff}^{(l)} \to \Lambda^{l-1} S_{\rm eff}^{(l)}, \tag{9}$$

under the transformation (8), provided that we write down all factors of δ_{st} explicitly even though it is just the Kronecker δ symbol. $S_{\rm eff}^{(l)}$ is said to have the conformal weight[5] $l-1$.

(ii) If we set the background field B equal to zero, and the spin connection ω constructed from the Christoffel symbol Γ equal to the gauge connection A^M, then the action (1) reduces to that of an $N=1$ supersymmetric sigma model.[1] If the background is Ricci flat, such models are known to be finite to all orders in the perturbation theory.[6] This fact may be used to obtain nontrivial constraints on the structure of the ultraviolet-divergent terms in the present model.

(iii) Under a field redefinition,

$$\psi \to U(X)\psi \equiv \exp[iT^M\theta^M(X)]\psi, \tag{10}$$

the action (1) is mapped into a similar expression with the A_i^M replaced by $A_i'^M$, given by

$$A_i'^M T^M = iU(X)\partial_i U^{-1}(X) + U(X)A_i^M T^M U^{-1}(X). \tag{11}$$

Hence the theories described by the gauge potentials A^M and A'^M are identical at the classical level. However, since A couples to chiral fermions, the gauge symmetry given by (11) is anomalous. As a result, we do not expect the β functions of the theory to be invariant under this symmetry. As we shall see, this indeed happens at the two-loop level, and is responsible for the appearance of the Chern-Simons term (which is not gauge invariant by itself) in the equations of motion.

(iv) Since we are working in the light-cone gauge, we have effectively assumed from the beginning that the x^0 and the x^1 directions are flat, and none of the dynamical fields acquire any vacuum expectation values in these directions. This causes us to lose some information from the equations of motion. This is best illustrated by an example. Consider, for example, Einstein's equation in the presence of a background electromagnetic field $F_{\mu\nu}$:

$$R_{\mu\nu} - \tfrac{1}{2}G_{\mu\nu}R = -8\pi G(F_{\mu\rho}F_\nu{}^\rho - \tfrac{1}{4}F_{\rho\sigma}F^{\rho\sigma}g_{\mu\nu}). \tag{12}$$

Since the 0 and 1 directions are flat, and F does not acquire any vacuum expectation values in these directions, the component of Eq. (12) in the (0,0) or (1,1) direction is

$$R = -4\pi G F_{\rho\sigma}F^{\rho\sigma}. \tag{13}$$

Equation (12) may then be written as

$$R_{\mu\nu} = (-8\pi G)F_{\mu\rho}F_\nu{}^\rho. \tag{14}$$

As we shall see, it is exactly equations of the type (14), without the $g_{\mu\nu}$ terms, that shall be obtained by the vanishing of the β function in the model described by Eq. (1).

(Alternatively, in string theory, the graviton field equations without the $g_{\mu\nu}$ terms may be interpreted as linear combinations of the graviton and the dilaton field equations.)

(v) In the present analysis we effectively assume the dilaton field to be constant throughout space, in which case it may be absorbed into the gauge coupling constant.[7] As a result, it will never appear explicitly in the analysis.

We may now proceed to analyze the equations for the vanishing of the β function at the one-loop level. They may be easily derived from Eq. (5) and are[1]

$$R_{il} + S_{ikm}S_l{}^{km} = 0, \tag{15}$$

$$D^j S_{ijl} = 0, \tag{16}$$

$$D^k F_{kl}^M - f^{MNP}A^{Nk}F_{kl}^P - S^{ij}{}_l F_{ji}^M = 0, \tag{17}$$

where R_{il} is the Ricci tensor.

After suitable normalization of the various fields, these equations may be shown to be identical to the equations of motion derived from the string theory (or the limiting field theory)[8] if we ignore all terms with conformal weight larger than 0 for Eqs. (15) and (16), and all terms with conformal weight larger than 1 for Eq. (17). Note that the last term on the left-hand side of Eq. (17) appears from the term in the effective action involving the Chern-Simons term for the gauge field, although it appears in the equations of motion in a gauge-covariant fashion.

Next, we turn to the two-loop calculation. In order to simplify the analysis, we shall only evaluate the contribution to the renormalization of the operators involving the X^i fields. Furthermore, we shall calculate this contribution only in the presence of background gauge fields. As a result, all terms involving B_{ij} and Γ_{ijk} are lost. Terms involving only the gravitational fields, however, may be recovered up to terms proportional to the Ricci tensor by demanding that the two-loop counterterm must vanish when we set $B=0$, the spin connection $\omega=A$, and $R_{ij}=0$. On the other hand, any term involving the Ricci tensor, or its derivatives, may be ignored in writing down the criteria for the vanishing of the β function if we ignore all terms with conformal weight larger than 1. This may be seen from Eq. (15), which says that if $B=0$, R_{ij} must be equal to terms with conformal weight >0 at the zero of the β function. When substituted into the two-loop effective action, this gives terms with conformal weight >1. Similarly, any term in the two-loop effective action proportional to the left-hand side of Eq. (17) may be set equal to zero in this approximation.

In order to simplify the analysis further, we shall assume that the background gauge field is Abelian in nature. This will cause the loss of all the cubic and quartic terms involving the gauge fields from the effective action. We now calculate the two-loop contribution to the effective action using the background-field method.[9] In doing this calculation we always first calculate the fermionic loop integral. We use the exact result[10] for the fermionic determinant in the presence of arbitrary background vector field a_α, coupling to a right-handed Weyl fermion. The contribution to the effective action for the a_α field from this determinant is given by

$$(8\pi)^{-1} a_\alpha \left\{ g^{\alpha\beta} - (g^{\alpha\alpha'} + \epsilon^{\alpha\alpha'}) \frac{\partial_{\alpha'}\partial_{\beta'}}{\partial^2} (g^{\beta'\beta} - \epsilon^{\beta'\beta}) \right\} a_\beta. \tag{18}$$

[Although the coefficient of the $g^{\alpha\beta}$ term in (18) is ambiguous,[10] it may be fixed by demanding that if a_α couples to a right-handed, as well as a left-handed Weyl fermion, the total contribution to the effective action should be gauge invariant.]

Since (18) does not have any ultraviolet divergences, we always get single poles in ϵ, and the evaluation of the graphs is straightforward. The total contribution to the effective action at the two-loop order is given by

$$\left(i \int \frac{d^2 l}{(2\pi)^2 (l^2 + i\epsilon)} \right) \tfrac{1}{16} [\{ F^M_{ki} F^{M}{}_l{}^i \partial_\alpha X^k \partial^\alpha X^l + \tfrac{1}{2} D^i (A_i^M F^M_{lk} + A_l^M F^M_{ki} + A_k^M F^M_{il}) \epsilon^{\alpha\beta} \partial_\alpha X^l \partial_\beta X^k \} - \{ {}^{A \to \omega}_{F \to R} \} + O(B) + O(A^2\omega) + O(A^3) + O(\omega^3)]. \tag{19}$$

Note the appearance of the Abelian Chern-Simons term in the above expression, which appears because of the chiral anomaly in two dimensions. The normalization has been chosen as $\mathrm{tr}(T^M T^N) = \delta^{MN}$.

We may now compute the two-loop β function in this model using the method of Friedan,[11] and write down the condition for the vanishing of the β function. Equations (15) and (16) are replaced by

$$R_{il} + S_{ikm} S_l{}^{km} + \tfrac{1}{8} (F^M_{ki} F^{Mk}{}_l - R_{imnp} R_l{}^{mnp}) = 0, \tag{20}$$

$$D^i [S_{ilk} + \tfrac{1}{16} \{ (A_i^M F^M_{lk} + A_l^M F^M_{ki} + A_k^M F^M_{il}) - ({}^{A \to \omega}_{F \to R}) \}] = 0, \tag{21}$$

with, of course, the error terms indicated in Eq. (19), as well as terms with conformal weight > 1. The appearance of the R^2 term in Eq. (19) has been predicted.[12]

Writing down the explicit factors of κ and g absorbed in S and F, respectively, using the relation $g^2 \sim \kappa^2/\alpha'$ and the fact that we have set $\alpha' = \frac{1}{2}$ in the present analysis, and properly normalizing F and S, one may show that Eqs. (20) and (21) are identical to the equations of motion for the graviton and the antisymmetric tensor fields, as derived from string theory.

Thus the present analysis indicates that there is an exact correspondence between the equations of motion of the string theory, and the condition for the vanishing of the β function in the model described by Eq. (1). Before concluding I wish to make the following remarks.

(i) In this analysis, I have not considered the equations of motion for the dilaton field. Since there are only three independent dimension-two operators in the model described by Eq. (1) that are not related to each other by a supersymmetry transformation, I cannot expect to get the equations of motion for the dilaton field by looking at the conformal invariance of the model. There is, however, another nontrivial constraint which must be satisfied by the models of this kind,[13] namely, that the correlation functions of the $++$ and the $--$ components of the energy-momentum tensor must remain unrenormalized from their free-field values. This constraint may reproduce the equations of motion for the dilaton field.

(ii) The current approach may also provide a way to tackle loop corrections in string theories. The higher-loop amplitudes in string theories correspond to formulating the string theory on world surfaces of nontrivial topology. However, the β function of the sigma model usually depends on the local properties of the manifold. Hence, the constraints imposed on the background fields by demanding the vanishing of the β function are expected to remain valid even when we include higher-loop corrections in the string theory.

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Note added.—After submitting this paper for publication, I came to know of a related paper by Callan *et al.*[14] which discusses issues similar to the ones discussed in this paper. My conjecture that the dilaton field equation may be obtained by looking at the correlation function $\langle T_{++}(\sigma,\tau) T_{++}(\sigma',\tau') \rangle$ and demanding that it should remain unrenormalized from its free-field value (or, equivalently, that the central charge of the Virasoro algebra should remain unrenormalized from its free-field value) has been shown to be correct in this paper.

[(a)]Present address: Stanford Linear Accelerator Center, Stanford University, Stanford, Cal. 94305.

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