Chiral-Symmetry Breaking in 2+1 Dimensions

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Chiral-symmetry breaking in (2+1)-dimensional QED is studied in the many-flavor limit. Analytical and numerical solutions of the Dyson-Schwinger equation are found. Substitution of the symmetry-breaking solution in the composite-operator effective potential indicates that it is favored over the symmetric solution. Improvements and possible extensions of the analysis are discussed.

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During the past decade, various attempts have been made to understand how global symmetries are realized in strongly interacting gauge theories. In four-dimensional QCD, it has been shown that vectorlike global symmetries are not spontaneously broken.[1] On the other hand, if QCD is the correct theory of strong interactions, then the axial-vector-like (chiral) symmetries are evidently broken, and several attempts have been made to demonstrate this result theoretically. The various analyses have used effective-potential methods,[2] lattice techniques,[3] large-N approximations,[4] anomaly conditions,[5] and a feasibility argument involving eigenvalue densities of Dirac operators.[6]

There is strong evidence that spontaneous chiral-symmetry breaking (CSB) in QCD is intimately connected with the confinement of color. The mass scale M associated with CSB is certainly of the same order as the confinement scale Λ at which the theory becomes strongly coupled. While there is some evidence that M may be larger than Λ,[2,3] their ratio is close to unity. Indeed, anything else would be quite strange in a theory without a small dimensionless parameter. In other theories with global chiral symmetries, however, such as dynamical theories of electroweak symmetry breaking, a large hierarchy $M \gg \Lambda$ might be contemplated and could lead to interesting consequences.[7,8]

In this Letter we describe an analysis of chiral-symmetry breaking in (2+1)-dimensional QED (QED_3),[9] a superrenormalizable theory having a dimensionful coupling constant e. When treated in a $1/N$ approximation, the model is tractable, and it has some of the features of four-dimensional theories. Solutions with spontaneous CSB can be sought, and the question of whether they are energetically preferred to the symmetric solution can be answered. Since the theory contains a small parameter $1/N$, a hierarchy between the CSB scale and the inherent mass scale $\alpha = e^2$ of the theory is possible. It will be seen that a hierarchy does exist in this model, but that it is the inverse of that contemplated above in four dimensions. The numerical analysis that we turn to in the end could well serve as a prototype for similar treatments of four-dimensional gauge theories.[8]

The Lagrangean of the model is

$$L = \sum_{i=1}^{N} \bar{\psi}_i (i\partial\!\!\!/ - eA\!\!\!/)\psi_i - \tfrac{1}{4}F_{\mu\nu}^2, \tag{1}$$

where ψ is taken to be a four-component spinor. The reason for our adopting a four-component formalism instead of a two-component one is that the gamma matrices γ^3 and γ^5 which anticommute with $\gamma^0, \gamma^1, \gamma^2$ can be used to define a global $U(2N)$ chiral symmetry of L. Even though in 2+1 dimensions a single two-component mass term is odd under P and T,[10] a four-component mass term is even. But such a term breaks the chiral symmetry to $U(1)\otimes U(1)\otimes SU(N)\otimes SU(N)$. This dynamical mass generation leads to the spontaneous breaking of a global symmetry. It would also be interesting to discuss the role of a parity-nonconserving mass term in spontaneous CSB. This type of fermion mass, which is related to a Chern-Simons term for the gauge field,[11–13] will be discussed elsewhere.

Massless QED_3 contains infrared divergences which cause the breakdown of the perturbation expansion.[12,14] It might be imagined that these divergences are removed only by spontaneous CSB.[15] That is what happens, for example, in the two-dimensional Gross-Neveu model.[16] In three-dimensional theories, however, it has been argued generally[12] and shown explicitly in the $1/N$ approximation[14] that mass generation is not necessary to produce an infrared-finite theory. When treated nonperturbatively, these theories can sensibly describe interacting massless particles. Three-dimensional QED in the $1/N$ approximation is such a theory. Even though spontaneous chiral-symmetry breaking is not forced by infrared divergences, it could nevertheless be preferred by the theory. We shall argue that this is the case.

To leading order in the $1/N$ expansion ($N \to \infty$ with $\alpha = e^2 N$ fixed), only the photon propagator is corrected from its free field form. In Landau gauge it takes the form

$$D_{\mu\nu}(p) = \frac{\delta_{\mu\nu} - p_\mu p_\nu / p^2}{p^2[1+\Pi(p)]},$$

where $\Pi(p)$ is given by the one-loop vacuum-polarization graph. With massless fermions, it is $\Pi(p) = \alpha/8p$. The infrared finiteness of the Green's functions of the theory then follows from the fact that, for $p \ll \alpha/8$, the photon propagator behaves like $1/p$ rather than $(1/p^2)$.[14] The interaction strength of the theory is characterized by the running coupling constant $\bar{\alpha} = \alpha/\{8p[1+\Pi(p)]\}$. In the ultraviolet it falls rapidly, while in the infrared it approaches the infrared-stable fixed point $\bar{\alpha} = 1$. The infrared well being of the massless theory can be traced to the existence of this fixed point.

Our study of chiral-symmetry breaking begins with the Dyson-Schwinger gap equation. In terms of the inverse Euclidean propagator $S_F'^{-1} = -\not{p}[1+A(p)]+\Sigma(p)$, it takes the following form to lowest nontrivial order in the $1/N$ approximation:

$$-\not{p}A(p)+\Sigma(p)=\frac{\alpha}{N}\int\frac{d^3k}{(2\pi)^3}D_{\mu\nu}(p-k)\frac{\gamma^\mu\{\not{k}[(1+A(k)]+\Sigma(k)\}\gamma^\nu}{k^2[1+A(k)]^2+\Sigma(k)^2}. \tag{2}$$

While $\Sigma(p)$ must be determined self-consistently by this equation, the wave-function renormalization $A(p)$ will be generated perturbatively in $1/N$. We therefore anticipate that $A(p)$ will be of order $1/N$ and drop it to leading order, focusing on the equation for $\Sigma(p)$. We shall return to a discussion of the reliability of this approximation. In general, $D_{\mu\nu}(p-k)$ will depend on $\Sigma(p)$, so that one is still faced with a set of coupled integral equations. For momentum scales large compared to $\Sigma(p)$ itself, however, Σ can be set equal to zero in $D_{\mu\nu}(p)$, so that $\Pi(p)$ is correctly given by $\alpha/8p$. After angular integration, the equation for $\Sigma(p)$ then takes the form

$$\Sigma(p)=\frac{\alpha}{2\pi^2Np}\int_0^\infty k\,dk\,\frac{\Sigma(k)}{k^2+\Sigma(k)^2}\ln\left[\frac{k+p+\alpha/8}{|k-p|+\alpha/8}\right]. \tag{3}$$

A nonzero solution $\Sigma(p)$ of this equation must compensate the explicit factor of $1/N$ on the integral. One must then examine the $1/N^2$ and higher corrections to this equation to check that they are not equally as large. The relative size of $\Sigma(p)$ and $\alpha/8$ is also of importance. The ratio must be small in order that Eq. (3) be reliable, and the question then is just how small? How large is the hierarchy, and how does it depend upon $1/N$? Equation (3) is valid for momentum p large compared to $\Sigma(p)$ itself. For momentum $k\sim\Sigma(k)$ in the integral, where the nonlinearity becomes important, the equation is not reliable in detail. We nevertheless retain $\Sigma(k)$ in the denominator as an infrared cutoff and a qualitative measure of the nonlinear structure of the gap equation. The expected form of the nonzero solution for $\Sigma(p)$ should begin to fall once $p\gg\Sigma(p)$ and then damp even more rapidly once $P\gg\alpha/8$, where the superrenormalizability of the theory is exhibited.

Analytic study of Eq. (3) is facilitated by breaking the integral into two pieces and expanding the logarithm:

$$\Sigma(p)=\frac{\alpha}{\pi^2Np}\int_0^p k\,dk\,\frac{\Sigma(k)}{k^2+\Sigma(k)^2}\left[\frac{k}{p+\alpha/8}+O\left[\frac{k}{p+\alpha/8}\right]^3\right]$$
$$+\frac{\alpha}{\pi^2Np}\int_p^\infty k\,dk\,\frac{\Sigma(k)}{k^2+\Sigma(k)^2}\left[\frac{p}{k+\alpha/8}+O\left[\frac{p}{k+\alpha/8}\right]^3\right]. \tag{4}$$

For both large p and small p (relative to $\alpha/8$), asymptotic forms may be found by keeping only the first term in each integral. The entire integral equation can then be converted into a second-order differential equation,

$$\frac{d}{dp}\left[\frac{d\Sigma(p)}{dp}\frac{p^2(p+\alpha/8)^2}{2p+\alpha/8}\right]=-\frac{\alpha}{\pi^2N}\frac{p^2\Sigma(p)}{p^2+\Sigma(p)^2}. \tag{5}$$

In the limit $p\ll\alpha/8$, but $p\gg\Sigma(p)$, this equation takes the simple linear form

$$\frac{d}{dp}\left[p^2\frac{d}{dp}\Sigma(p)\right]=-\Sigma(p)\left[\frac{8}{\pi^2N}\right].$$

Only if the result $\Sigma(p)\ll\alpha/8$ emerges from the full nonlinear equation, however, will this linear equation correctly describe $\Sigma(p)$ in any regime at all. Its solution is $\Sigma(p)=Ap^a$, where $a=-\frac{1}{2}\pm\frac{1}{2}(1-32/\pi^2N)^{1/2}$. For large N, the two solutions are $\Sigma_1(p)\sim p^{-8/N\pi^2}$ and $\Sigma_2(p)\sim p^{-1+8/N\pi^2}$, both falling with increasing p. It is also interesting to consider the case $N<32/\pi^2$ even though the $1/N$ expansion then becomes a bit suspicious (our numerical solutions have so far, in fact, been limited to this range). The solution then falls as $k^{-1/2}$ times a function that oscillates in $\ln(p)$. Whether enough space exists between $\Sigma(p)$ and $\alpha/8$ to see any oscillations depends again on the solution to the full nonlinear equation. We shall return to this point after presenting the numerical results. The solutions found here give small contributions to the higher-order terms in Eq. (4) in the limit $p\ll\alpha/8$. This justifies dropping them initially.

Equation (5) can also be solved in the asymptotic regime $p\gg\alpha/8$. There are two series solutions of the form $\Sigma_A(p)=(A/p^2)\{1+a\alpha/p+\dots\}$ and $\Sigma_B(p)=B\{1+b\alpha/p+\dots\}$. The second solution is effectively a hard mass for the fermion and therefore does not correspond to spontaneous CSB. The first solution $\Sigma_A(p)$ falls as expected with p. The coefficient a is given by $a=-\frac{1}{8}-\frac{2}{3}N\pi^2$. $\Sigma_A(p)$ gives a contribution to the upper integral of the integral equation (4) that is suppressed by a factor α/π^2Np

relative to $\Sigma_A(p)$ itself. It is, therefore, the lower integral that the solution must saturate, but this requires more information than the asymptotic series solution for $p \gg \alpha/8$. A function that solves the truncated lower integral equation is

$$\Sigma_A(k) = \left[\frac{D}{k^2 + k\alpha/8}\right]\left[\frac{k}{k+\alpha/8}\right]^{8/\pi^2 N}.$$

It agrees with the first term in the asymptotic series for $p \gg \alpha/8$ and it even agrees with the second term for large N. There is no reason to believe that it is completely reliable for $p \sim \alpha/8$, however, since the other term in Eq. (5) cannot be neglected there. It is only the asymptotic behavior $\Sigma(p) = A/p^2$ that has been reliably obtained.

Although sensible qualitative behavior can be obtained analytically in extreme limits, the full nonlinear integral equation (3) is far too complicated for a general analytic study. In order to find the general solution and to answer the questions posed above, we turn to a numerical study of CSB in QED_3.

Before we present these results, the effective-potential formalism that naturally accompanies the Dyson-Schwinger gap equation is described. An effective potential as a functional of $\Sigma(p)$ can be constructed following Cornwall, Jackiw, and Tomboulis[17] and Peskin.[18] To discuss the solution to the gap equation (3), it suffices to carry the computation of the potential to second order in the $1/N$ expansion. If we continue to neglect wave-function renormalization, the zeroth-order potential, corresponding to a single fermion loop, is

$$V_0 = \frac{N}{\pi^2}\int_0^\infty p^2\,dp\left[\frac{2\Sigma(p)^2}{p^2+\Sigma(p)^2} - \ln\left[1+\frac{\Sigma(p)^2}{p^2}\right]\right], \tag{6}$$

where a subtraction has been performed to normalize V_0 to zero at $\Sigma(p)=0$. The next order term corresponds to the emission and reabsorption of a photon from the fermion loop. After angular integration, it takes the form

$$V_1 = -\frac{\alpha}{2\pi^4}\int_0^\infty p\,dp\,\frac{\Sigma(p)}{p^2+\Sigma(p)^2}k\,dk\,\frac{\Sigma(k)}{k^2+\Sigma(k)^2}\ln\left[\frac{k+p+\alpha/8}{|k-p|+\alpha/8}\right]. \tag{7}$$

Note that V_1 also vanishes when $\Sigma(p)=0$.

If the potential $V = V_0 + V_1$ is extremized by setting its functional derivative with respect to $\Sigma(p)$ equal to zero, the Dyson-Schwinger gap equation (3) is obtained. The value of the potential at the extremum is a measure of the stability of the CSB solution. Substituting the Dyson-Schwinger equation back into the potential yields

$$V_{\text{ext}} = \frac{N}{\pi^2}\int_0^\infty p^2\,dp\left[\frac{\Sigma(p)^2}{p^2+\Sigma(p)^2} - \ln\left[1+\frac{\Sigma(p)^2}{p^2}\right]\right]. \tag{8}$$

One can easily see that this expression is negative by noting that $x/(1+x) - \ln(1+x)$ is negative for all positive x. Therefore, if a symmetry-breaking solution can be found, it will always be preferred to the symmetric one, if all the higher-order corrections can be shown to be small.

Turning now to the numerical study, the Dyson-Schwinger equation (3) has been solved self-consistently. Nonzero solutions for $\Sigma(p)$ have been found. They have the expected qualitative behavior discussed above. This study has so far been limited to values of N from 1 to 3, including intermediate nonintegral values. To achieve sensitivity to the shape of the integrand, the size of the integration grid has to be smaller than $\Sigma(0)$, which falls sharply with N. Therefore, the computer time rises rapidly with increasing N, and we decided to stop at $N=3$ for the time being. There appears to be no reason, however, why solutions should not continue to exist for large, even arbitrarily large, values of N. As a function of p, $\Sigma(p)$ starts out at some constant value $\Sigma(0)$ and begins to fall monotonically once $p \gg \Sigma(0)$. The ratio $\Sigma(0) \times (\alpha/8)^{-1}$ is small even for $N=1$ and drops rapidly as N is increased. The falloff of $\Sigma(p)$ with p continues for $p \gg \alpha/8$, although no qualitative change in the rate of fall as p passes through $\alpha/8$ has so far been discerned. The method of solution makes use of an ultraviolet cutoff $\Lambda \gg \alpha/8$ on the integral, and the solutions are quite insensitive to Λ. A plot of $\Sigma(p)$ vs p for $N=2.6$ is shown in Fig. 1.

The ratio $\Sigma(0)(\alpha/8)^{-1}$ is a measure of the hierarchy between the chiral-symmetry—breaking scale and the fundamental scale in the theory. Table I shows the numerical results for $-\ln[\Sigma(0)(\alpha/8)^{-1}]$ plotted against N. It seems clear that the falloff with N is at least exponential, but it is probably best to obtain points at some higher values of N before a fit is attempted. The gap between $\Sigma(0)$ and $\alpha/8$ is large but not so large that the oscillatory solutions found analytically in the region $\Sigma(p) \ll p$

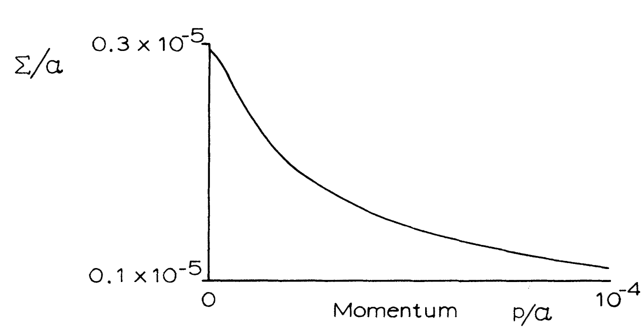

FIG. 1. The fermion self-energy $\Sigma(p)$ as a function of momentum for $N=2.6$ fermion flavors.

TABLE I. The fermion self-energy at zero momentum $\Sigma(0)$ as a function of the number of flavors N.

N	$-\ln\left[\frac{\Sigma(0)}{\alpha/8}\right]$
1	2.3
1.2	2.9
1.4	3.6
1.6	4.3
1.8	5.1
2.0	6.1
2.2	7.2
2.4	8.6
2.6	10.7
2.8	13.8
3.0	19.5

$\ll \alpha/8$ will obviously play a role. No numerical evidence of oscillations is seen at all.

Our use of the $1/N$ expansion, especially for the small values of N explored numerically so far, is surely open to criticism. It must be checked that, with the solutions found to Eq. (3), the higher-order terms in the $1/N$ expansion really are small corrections.[19]

One such higher-order effect is the wave function renormalization $A(p)$. A one-dimensional equation for $A(p)$, analogous to Eq. (3) for $\Sigma(p)$, can be extracted from Eq. (2). $A(p)$ can then be computed perturbatively in $1/N$ by starting with $A=0$ on the right-hand side. The solution for $\Sigma(p)$ when $A=0$ can also be used on the right-hand side. We have checked numerically that $A(p)$ does indeed fall as $1/N$ for fixed p, for values of N up to 3. This no doubt continues for arbitrarily high values of N. The $1/N$ corrections to $\Sigma(p)$ can be computed by substitution of results like this back into the corrected equation for $\Sigma(p)$. Since $A(p)$ is gauge dependent, it must be combined with the vertex correction before one tries to estimate the effect on $\Sigma(p)$. For small k, $A(k)$ behaves like $(1/N)\ln(k/\alpha)$, and the factor $\ln(k/\alpha)$, when integrated over, could generate a power of N to neutralize the $1/N$ suppression. However, the $\ln(k/\alpha)$ is canceled by a similar term from the vertex correction, and therefore the combinations of these effects will give a correction of order $1/N$ to $\Sigma(p)$ for all values of p.

Other higher-order terms in the $1/N$ expansion must also be examined. Another important question is that of gauge dependence. It must be shown that, although particular details may change with gauge, the essential conclusions remain intact. Further study of the effective potential and the stability of the chiral-symmetry—breaking solutions is also warranted. Finally the numerical techniques developed here might profitably be applied to some realistic four-dimensional theories, such as those that could underlie electroweak symmetry breaking.

We conclude with a remark about the analysis of Pisarski.[9] He computed the ratio $\Sigma(0)/\alpha$ by assuming that $\Sigma(p)$ was a constant and cutting the integral off at a momentum on the order of α. He then concluded that chiral-symmetry—breaking solutions exist, that they are preferred energetically, and that for large N, $\Sigma/\alpha = ce^{-\pi^2 N/8}$, where c is of order 1. Even though his initial assumption was in disagreement with our analytical and numerical results for $\Sigma(p)$ as a function of p, his conclusion that chiral symmetry spontaneously breaks in QED_3 is in agreement with ours.

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[19]The special concern here stems from the fact that the solutions must be N dependent in order to balance the $1/N$ on the right-hand side of Eq. (3). Further seemingly subdominant corrections might then also build up to become N independent.