Symmetric Kicked Self-Oscillators: Iterated Maps, Strange Attractors, and Symmetry of the Phase-Locking Farey Hierarchy

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The stroboscopic map of symmetric self-oscillators driven by pulses of alternating sign is constructed, for the regime of strong relaxation, by means of the so-called phase-transition curve. An ordering for the symmetries of the phase-locked orbits is found which is conjectured to be universal. Both results are illustrated by an exactly solvable example of such a system. When the approximation fails, unusual period doubling from symmetric orbits and strange attractors of two-dimensional structure are encountered.

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Nonlinear oscillators periodically excited by pulses of constant amplitude, short enough to be simulated by δ functions, appear in several experimental and theoretical situations in physics,[1] chemistry,[2] biology,[3] and other sciences.[4] In the strong-relaxation regime, the dynamics of these models is very accurately described by a one-dimensional Poincaré map which can be constructed if we know the phase-transition curve (PTC),[5] i.e., the effect on the phase of the oscillation produced by an isolated pulse. In this way the general behavior (phase-locking structure, transition to chaos, etc.) has been (and is being) widely studied and understood. However, and despite the many possible applications, no work has been done in order to apply this method to the treatment of more general pulsatile driving forces. Especially interesting is the case which consists of a sequence of pulses of equal amplitude and equally spaced but of alternating sign. These forces can appear in electronics, biology (polarizing and depolarizing effect of stimuli), far-from-equilibrium chemical reactions (periodic input and annihilation of reactants), and so on. The purpose of this Letter is threefold. Firstly, we show how the presence of a symmetry allows us to use the PTC for the dynamical description of a nonlinear oscillator driven by alternating forces. Secondly, we numerically check the method in an exactly solvable model whose behavior under equal-sign periodic δ forces has been previously studied.[6] In this model, we also find, for certain parameter values, the unusual presence of period doubling from a symmetric solution without a symmetry-breaking precursor. Finally, we give a universal ordering of the symmetry of the periodic solutions which is related to the Farey tree construction.

Let us consider a driven self-oscillating system described by

$$\dot{x} = y; \qquad \dot{y} = f(x,y) + V_E \sum_{n=0}^{\infty} (-1)^n \delta(t - nT_E/2), \tag{1}$$

where V_E and $\omega_E = 2\pi/T_E$ are the amplitude and frequency of the external force and $f(-x,-y) = -f(x,y)$. When $V_E = 0$, (1) is assumed to have a limit cycle which is symmetric, whereas if $V_E \neq 0$ (1) is invariant under $(x,y,t) \rightarrow (-x,-y,t+T_E/2)$. It is easy to show that the Poincaré (stroboscopic) map P can be written as the second iteration of another transformation $\tilde{P}$[7]:

$$P = \tilde{P}^2 = \tilde{P} \circ \tilde{P}; \qquad \tilde{P}(x,y) = -I \circ P_{t_0}^{t_0 + T_E/2}(x,y), \tag{2}$$

with $P_{t_0}^{t}(x(t_0),y(t_0)) = (x(t),y(t))$ being a solution of (1) and I the 2×2 unit matrix. If at $t = t_0$ a pulse is applied, $P_{t_0}^{t+T_E/2}$ becomes the Poincaré map of (1) without the factor $(-1)^n$ in the external force. In the strong-relaxation regime the stroboscopic map degenerates into a transformation of the limit cycle on itself.[6,8–10] Moreover, the map can be written now as a function which only depends on the phase of the oscillation just before each pulse. Thus

$$P_{t_0}^{t_0+T_E/2}(x,y) \rightarrow Q_{t_0}^{t_0+T_E/2}(\phi_0) = g(\phi_0, V_E) + \omega_0 T_E/2, \tag{3}$$

where ϕ_0 is the mentioned phase, g is the PTC which takes into account the change of the phase due to an isolated pulse, and ω_0 is the proper frequency of the oscillator.

As a result of the symmetry of the limit cycle the $-I$ operator in (2) is represented, in this approximation, by a shift of π in the phase ϕ. Thus, the map $Q = \tilde{Q}^2$ which describes the dynamics of the system with the alternating pulses is obtained from

$$\tilde{Q}_{T_E,V_E}(\phi) = -I \circ Q_{t_0}^{t_0+T_E/2}(\phi) = g(\phi, V_E) + \omega_0 T_E/2 + \pi = g + \omega_0(T_E + T_0)/2. \tag{4}$$

Periodic solutions of (1) or, equivalently, periodic points of P can be now investigated by means of Q and $\tilde{Q}$. Each q-periodic orbit of Q or $\tilde{Q}$ is characterized by its winding number p/q when $f^q(\phi)=\phi+2\pi p$ ($f=Q$ or $\tilde{Q}$). It is easy to show that every $2q$-periodic orbit $P/2q$ of $\tilde{Q}$ is a q-periodic asymmetric orbit $(p-q)/q$ of Q. Analogously, every $(2q+1)$-periodic orbit $p/(2q+1)$ of $\tilde{Q}$ is a $(2q+1)$-periodic symmetric orbit $(2p-2q-1)/(2q+1)$. On the other hand, the chain rule guarantees that the stability of both orbits (those corresponding to Q and $\tilde{Q}$) is the same. Thus, the phase-locking spectrum in the (T_E,V_E) parameter space of the oscillator, when it is forced by alternating pulses, can be related to the analogous one of the oscillator forced by pulses of constant sign. The correspondence can be realized in the following way: Take the phase-locking diagram for the second case, shift the T_E axis by an amount $\frac{1}{2}T_0$ and expand it by a factor of 2, and, finally, divide the even periodicities by 2.

As an example, we have plotted [Fig. 1(a)] the stability regions in the (T_E,V_E) plane of the periodic solutions of the previously studied model[6,10]

$$\ddot{x}+(4bx^2-2a)\dot{x}+b^2x^5-2abx^3+(\omega_0^2+a^2)x$$
$$=V_E\sum_{n=0}^{\infty}(-1)^{rn}\delta(t-nT_E/2^r), \quad (5)$$

where a and b are parameters and $r=0$, for comparison [Fig. 1(b)] with the regions in the same plane obtained for $r=1$ (alternating pulses). In the last case, the system has the required symmetry. The above-mentioned correspondence clearly appears in Figs. 1(a) and 1(b) for long values of T_E. However, for $T_E \lesssim T_0/2$ it is broken. This is because, for such values of the external frequency, the strong-relaxation approximation is not justified any more. In this regime, the map P is two dimensional and essentially different from the stroboscopic map of the system with $r=0$. The system shows, for these values of T_E, an interesting behavior of bifurcations. It is often reported in the literature that symmetric solutions do not bifurcate to period-doubling ones without a previous symmetry breaking. This behavior has recently[7] been well understood for "purely" dissipative forced oscillations and for *generic* one-parameter families of dynamical systems which can be described with two-dimensional maps, both having the symmetry of (1). However, our model, as well as all externally driven self-oscillators, does not belong to such a class.[11] In fact, there is a curve in the parameter space (see Fig. 1) at which period doubling from a symmetric solution takes place.[12] In Fig. 1(c) we have plotted the symmetric orbit near the bifurcation point and the period-doubled orbit after the bifurcation. The new orbit is asymmetric, which reflects the simultaneous occurrence of a symmetry-breaking bifurcation. At the end of the subsequent cascade of period-doubling bifurcations, strange attractors appear. In Figs. 1(c)–1(f) we show three magnifications of one of these attractors stroboscopically sampled. Its complicated structure is similar to those arising in two-

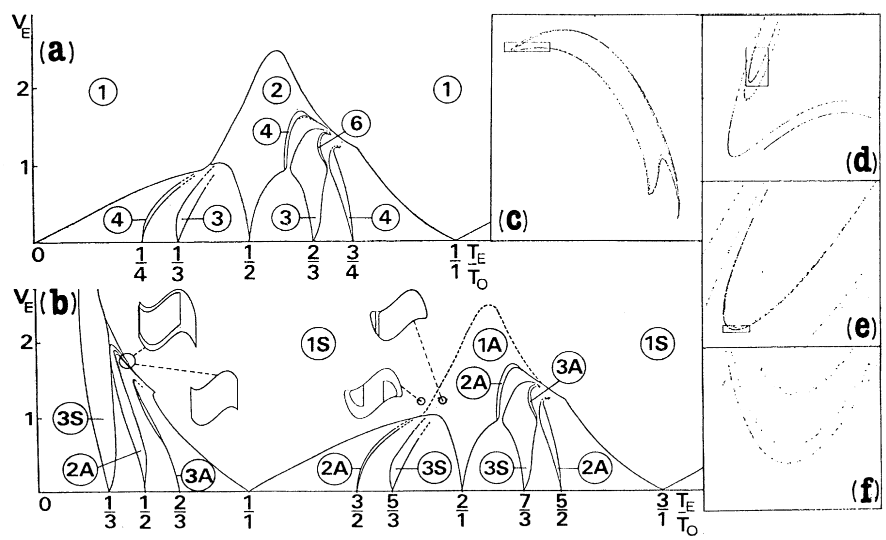

FIG. 1. Phase-locking diagrams in the plane (T_E,V_E) of the system (5) for $a=1.57079$, $b=15.7079$, and $\omega_0=1.57079$; (a) $r=0$, and (b) $r=1$. In the last case, period doubling from symmetric orbits and symmetry-breaking bifurcations are illustrated. Circled numbers in (a) and (b) and letters in (b) label the period and the symmetry of the corresponding region. (d)–(f) Successive enlargements of the strange attractor shown in (c) for $T_E=1.628$ and $V_E=1.695$; panel (c) is the set $(0.07,0.53)\otimes(-2.15,0)$ and each of panels (d)–(f) is drawn in the preceding one.

dimensional iterated maps.[13] This is another evidence for the failure of the one-dimensional description at these values of the parameters.

In the remaining part of this paper we establish a rule which allows us to know whether a given periodic orbit of a symmetric system is symmetric or asymmetric by knowing its rotation number. We claim that this rule is universal.

It is well known that several qualitative features of the phase-locking structure of driven nonlinear oscillators can be universally described with the aid of the so-called Farey sum.[14,15] Moreover, in a beautiful recent work by Feigenbaum,[16] universal quantitative predictions on the phase-locking structure have been obtained in a way which is also related to the Farey sum. Here we put the symmetry of the orbits in the same frame.

Let $\oplus$ denote the Farey sum operation between two rational numbers. Thus, if $r_1 = p_1/q_1$ and $r_2 = p_2/q_2$ are two such numbers, then $r_1 \oplus r_2 = (p_1+p_2)/(q_1+q_2)$. This operation has the following properties: if $r_1 = p_1/q_1$ and $r_2 = p_2/q_2$ satisfy the unimodularity condition $|p_1q_2 - p_2q_1| = 1$ then $r_3 = r_1 \oplus r_2$ satisfies the same condition with both r_1 and r_2. Unimodular numbers are also said to be "adjacent" and their Farey sum is called "mediant." All the rational numbers between 0 and 1 (and so all the phase-locking regions) can be organized in the so-called Farey tree. Starting with the numbers 0/1 and 1/1 we define the zeroth level of the tree as $1/2 = 0/1 \oplus 1/1$. The first level is obtained by Farey addition of all the previous numbers: $1/3 = 0/1 \oplus 1/2$ and $2/3 = 1/2 \oplus 1/1$. This construction continues recursively. At the nth stage, the nth level of the tree which contains 2^n members is obtained [see Fig. 2(a)]. Our main result here is the following: A given orbit is characterized by a rational winding number which is obtained by the Farey sum from two parents. If the two corresponding orbits are asymmetric, then the resulting one is symmetric; if only one parent is symmetric, then the orbit is asymmetric. That is all, because two symmetric parents cannot exist (they cannot be adjacent). The proof of this statement is not difficult and the details will be given elsewhere.

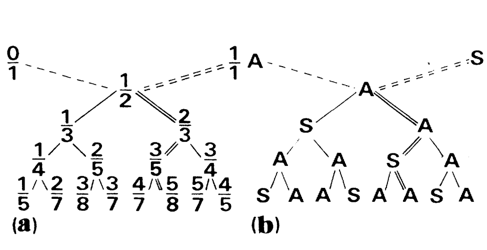

FIG. 2. (a) The first four levels of the Farey tree; the lines join "parents" with their respective "sons"; the path indicated by double lines passes through the successive approximants of the golden mean. (b) Symmetry labels of the orbits whose rotation number is the corresponding rational number in (a).

Figure 2(b) shows what the structure of the Farey tree looks like when the symmetry of the solution is considered. Symmetric orbits are labeled S and the asymmetric ones A. Observe that at each level of the tree the labels are arranged in sequences of groups AAS. Moreover, for a given level, say n, we have the following: (a) If $2^n/3 = p + \frac{1}{3}$, where p is an integer, then the sequence of characters is

$$(AAS \ldots p \text{ groups} \ldots AAS)A. \tag{6}$$

(b) If $2^n/3 = p + \frac{2}{3}$, then the sequence is

$$S(AAS \ldots p \text{ groups} \ldots AAS)A. \tag{7}$$

For the complete arrangement of labels at a given stage (say the nth) of the construction of the Farey tree, we found that (a) when $\sum_{r=0}^{n} 2^r/3 = p$ the sequence is $S(AAS \ldots p-1 \text{ groups} \ldots AAS)AA$; and (b) when $\sum_{r=0}^{n} 2^r/3 = p + \frac{1}{3}$ it is $AS(AAS \ldots p-1 \text{ groups} \ldots AAS)AA$.

At this point we want to make the following remarks: (i) The sequences (6) and (7) coincide with the sequence of characters associated with the orbits whose winding numbers correspond to the successive approximants of the golden mean ($\frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \ldots$) as can be seen in Fig. 2. This fact is consistent with Feingenbaum's result,[16] which shows that this number plays a role in the universal description of the phase-locking structure for highly iterated orbits. (ii) We can recognize the symmetry of an orbit from its winding number: If the sum of numerator and denominator is even the orbit is symmetric, otherwise it is asymmetric. (iii) It is known[14] that the phase-locking regions can, in general, be ordered according to their size by taking into account that the greatest of the regions localized between two adjacent ones is just the region whose winding number is the mediant. The two parents are greater than the daughter. Even though this rule works in typical forced oscillators, we have seen that the symmetry drastically modifies this behavior. In this case, a similar scheme can be constructed by replacing the unimodularity condition in the definition of adjacency by the following: $|p_1q_2 - p_2q_1| = 2$.

Finally, we want to point out that our results can be rigorously proved for the strong-relaxation regime. However, our experience with the model (5) indicates that they extend to the case at which this approximation fails. It is also interesting to note that, when varying V_E, the map Q (and $\tilde{Q}$) becomes noninvertible, and the phase-locking regions which persist have the same symmetry as in the invertible case. The route to chaos starts now always with a symmetry-breaking bifurcation when the original orbit is symmetric.[12]

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[11]In general, the divergence of the vector field in (1) is positive inside the limit cycle and is negative for large enough values of (x). These systems dissipate energy in some regions of the phase space and "create" it in others. Besides, there are two controlled external parameters.

[12]Symmetric period doubling corresponds to the crossing of a pair of eigenvalues of the linear part of $\tilde{P}$ through $\pm i$. This is not possible, obviously, for $\tilde{Q}$ because it is a *real* map on a *one-dimensional* manifold. Thus, in the regime of strong relaxation, such bifurcations are absent.

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