Dynamics on Ultrametric Spaces

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We present an exact solution to the problem of a random walk on an ultrametric space with arbitrary transition probabilities. The solution provides a description of fluctuation dynamics of systems with many degenerate states separated by a hierarchy of energy barriers. Dynamical behavior is found to depend on the scaling of barriers with ultrametric distance. We find a range of interesting behavior, from temperature-dependent power-law decay to the Kohlrausch law.

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Since Mezard *et al.*[1] showed that the space of ground states of the mean-field spin-glass possessed an ultrametric topology, it has been speculated that a number of other complex systems with highly degenerate, locally stable states exhibit this structure. These include physical systems such as glasses and spin-glasses,[2] proteins,[3] and "hard" combinatorial optimization problems such as the traveling salesman problem,[4] etc. Such systems appear to be widespread and in general relax very slowly, which suggests that the dynamics of such structures could have common features and thus are of general importance. Palmer, Stein, Abrahams, and Anderson[5] (PSAA) proposed a class of models of hierarchically constrained dynamics; the context in which hierarchies appeared there was different from that of Mezard *et al.*[1] Huberman and Kerszberg[6] used an approximate renormalization-group analysis to investigate diffusion in a simple, uniformly bifurcating space in the special case where successive barriers increased arithmetically. A number of other related special cases have been worked out as well.[7,8]

In this paper we present the exact, general solution for the dynamics of systems characterized by highly degenerate, locally stable states separated by energy barriers defining an ultrametric topology.

Consider the stochastic dynamics of a system with a countable space of states which evolves in time by fluctuation from state to state; transitions between states are thermally activated with rates determined by the (free) energy barriers separating the states. The space of states has the metric topology defined by barrier heights in the following sense: Let us rank the barriers in order of increasing magnitude $\Delta_1 < \Delta_2 < \Delta_3 < \ldots < \Delta_k < \ldots$; we will say that two states are separated by distance d if the (free) energy barrier for a transition between them is Δ_d. This problem can be formulated as a random walk on the space of states.

The structure of the space we will consider is presented in Fig. 1. The random walk occurs only on the top level, at points $0, 1, 2, 3, \ldots$. The total number of points equals 2^n, $n = 0, 1, 2, \ldots$. The ultrametric distance d between two points is given by the number of branches one must descend from the top level before the branches merge. The walker encounters barriers for any jump, and the size of the barriers depends on the distance traveled in the jump. The barriers may be arithmetically increasing, have a random distribution, and so on, depending on which model one chooses.

Let the probability of the particle being found at site i at time t be given by $P_i(t)$; hence, $\sum_{i=0}^{2^n-1} P_i(t) = 1$. Further, let the probability per unit time of jumping an ultrametric distance of 1 be given by ϵ_1; of jumping to *one particular* of the two sites an ultrametric distance 2 away, ϵ_2; of jumping to *one particular* of the four sites a distance 3 away ϵ_3, and so on. The dynamical equation governing the time evolution of $\mathbf{P}(t)$ is

$$\partial \mathbf{P}(t)/\partial t = \boldsymbol{\epsilon} \mathbf{P}(t). \qquad (1)$$

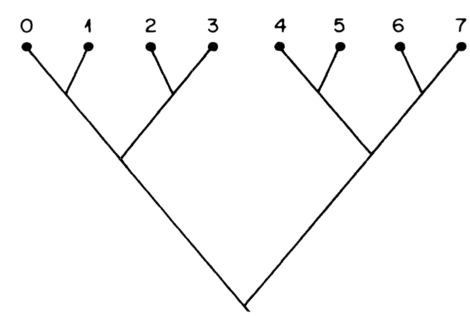

FIG. 1. A simple bifurcating ultrametric space. As measured from site 0, site 1 has an ultrametric distance of 1, sites 2 and 3 have an ultrametric distance of 2, and sites 4–7 have an ultrametric distance of 3.

The transition matrix $\boldsymbol{\epsilon}$ is

$$\epsilon = \left[\begin{array}{cc} \begin{array}{c|c} \begin{matrix}\epsilon_0 & \epsilon_1\\ \epsilon_1 & \epsilon_0\end{matrix} & E_2 \\ \hline E_2 & \begin{matrix}\epsilon_0 & \epsilon_1\\ \epsilon_1 & \epsilon_0\end{matrix}\end{array} & E_3 \\ E_3 & \begin{array}{c|c} \begin{matrix}\epsilon_0 & \epsilon_1\\ \epsilon_1 & \epsilon_0\end{matrix} & E_2 \\ \hline E_2 & \begin{matrix}\epsilon_0 & \epsilon_1\\ \epsilon_1 & \epsilon_0\end{matrix}\end{array} \\ & \vdots \end{array} \cdots \right],$$

where the block E_2 is a 2×2 matrix with all entries ϵ_2, E_3 is a 4×4 matrix, and E_k is a $2^{k-1}\times 2^{k-1}$ matrix with identical entries ϵ_k and $\epsilon_0 = -(\epsilon_1 + 2\epsilon_2 + 4\epsilon_3 + \ldots + 2^{n-1}\epsilon_n)$. Note that this matrix is of the Parisi form.[9]

We use for convenience the initial condition $P_0(0)=1$, $P_i(0)=0$ for all $i\geq 1$. We shall be interested in studying the behavior of both the autocorrelation function $P_0(t)$ and the average distance traveled in time t:

$$\langle d(t)\rangle = \textstyle\sum_k d(k,0)P_k(t), \qquad (2)$$

where $d(k,j)$ is the ultrametric distance between site k and site j. If the walker is at a unique site with probability one at time zero, the behavior of both of these quantities is independent of the starting site.

We now find the eigenvalues and eigenvectors for the general $2^n\times 2^n$ matrix $\boldsymbol{\epsilon}$. Because $\boldsymbol{\epsilon}$ is the transition matrix for a stochastic process, all of its eigenvalues but one will be negative. The one nonnegative eigenvalue is zero, with eigenvector $\mathbf{P}$ such that $P_l = 2^{-n}$ for all l. This corresponds to the stationary distribution to which the system evolves at long time. A little thought will convince the reader of the following assertions:

(1) The eigenvalue $\eta_0 = \epsilon_0 - \epsilon_1$ is 2^{n-1}-fold degenerate. Its eigenvectors can be constructed by partitioning of the vector elements into groups of two. Each eigenvector has nonzero entries in only *one* group of two, and the sum of the entries is zero. This eigenvalue measures the rate of escape from a single site.

(2) The eigenvalue $\eta_1 = \epsilon_0 + \epsilon_1 - 2\epsilon_2$ is 2^{n-2}-fold degenerate. The eigenvectors here correspond to all partitions of vector elements into groups of four such that only one group of four has nonzero entries. Of this group, the first two elements are equal, and the second two are their negatives. This eigenvalue corresponds to the rate of escape from a cluster of two neighboring sites.

(3) In general, the eigenvalue η_m $(0\leq m\leq n-1)$ is 2^{n-1-m}-fold degenerate and corresponds to the escape rate from a cluster of 2^m neighboring sites. Its value is

$$\eta_m = \epsilon_0 + \sum_{k=1}^{m} 2^{k-1}\epsilon_k - 2^m\epsilon_{m+1}. \qquad (3)$$

The eigenvectors are obtained by grouping the vector elements into groups of 2^{m+1}, with only one group having nonzero entries. The sum of all elements in this group is zero, with the first half of the elements in the group all equal and the second half their negatives.

(4) There are only two nondegenerate eigenvalues:

$$\eta_{n-1} = \epsilon_0 + \sum_{k=1}^{n-1} 2^{k-1}\epsilon_k - 2^{n-1}\epsilon_n,$$

which corresponds to an n-site hop, and $\eta_n = 0$.

In order to simplify the notation, we shall set $\lambda_m = -\eta_m$, so that $\lambda_0 > \lambda_1 > \lambda_2 > \ldots > \lambda_{n-1} > \lambda_n = 0$. Also, we set $a_k = 2^{k-1}\epsilon_k$ $(k\geq 1)$; a_k represents the probability per unit time of jumping an ultrametric distance k (i.e., to *any* of the 2^{k-1} sites a distance k from a given starting site). Then Eq. (3) becomes

$$\lambda_m = 2a_{m+1} + \sum_{k=m+2}^{n} a_k \quad (m < n-1),$$
$$\lambda_{n-1} = 2a_n. \qquad (4)$$

If hopping is thermally activated and if the particle encounters a barrier of size Δ_m in hopping a distance m, then $a_m = e^{-\Delta_m/T}$ at temperature T (we set the attempt frequency $\omega_0 = 1$). Note that with this definition of barriers there exists only one barrier Δ_m which separates a cluster of 2^{m-1} sites from the initial site. Barriers are then associated with the branching points in Fig. 1.

As an illustrative example, let us consider the case of eight sites $(n=3)$. The initial condition $P_0(0)=1$, $P_i(0)=0$ $(i\geq 1)$ leads to the solution

$$\mathbf{P}(t) = 2^{-3}\begin{bmatrix}1\\1\\1\\1\\1\\1\\1\\1\end{bmatrix} + 2^{-1}e^{-\lambda_0 t}\begin{bmatrix}1\\-1\\0\\0\\0\\0\\0\\0\end{bmatrix} + 2^{-2}e^{-\lambda_1 t}\begin{bmatrix}1\\1\\-1\\-1\\0\\0\\0\\0\end{bmatrix} + 2^{-3}e^{-\lambda_2 t}\begin{bmatrix}1\\1\\1\\1\\-1\\-1\\-1\\-1\end{bmatrix}, \qquad (5)$$

with the λ_k given by Eq. (4).

We are now ready to write down the exact solutions for $P_0(t)$ and $\langle d(t)\rangle$ in the general 2^n-site case:

$$P_0(t)=2^{-n}+\frac{1}{2}\sum_{m=0}^{n-1}\exp(-m\ln 2)\left[\exp(-2a_{m+1}t)\prod_{i=m+2}^{n}\exp(-a_i t)\right], \tag{6}$$

$$\langle d(t)\rangle=(n-1)-\sum_{m=0}^{n-1}\left[\exp(-2a_{m+1}t)\prod_{i=m+2}^{n}\exp(-a_i t)\right]+P_0(t). \tag{7}$$

The product term in Eqs. (6) and (7) is equal to 1 in the particular case $m=n-1$. We note that $P(t)$ corresponds to the relaxation function $q(t)$ of PSAA, and that in the limit $n\to\infty$ we may write

$$P_0(t)=\frac{1}{2}\sum_{m=0}^{\infty}w_m e^{-\lambda_m t}, \tag{8}$$

which has the same formal structure as Eq. (7) in PSAA (with $\lambda_m=\tau_m^{-1}$). A correspondence can therefore be drawn between certain special cases described here and certain of those in PSAA. Relaxation times for various levels in the PSAA model would then correspond to eigenvalues of the evolution operator, and their weights to their degeneracies. Physically, the faster modes correspond to escape from smaller clusters of states, and the slower modes to escape from larger clusters.

We can use Eqs. (6) and (7) to study various cases. The simplest one is that with a uniform barrier Δ at every branch point; that is, a jump of distance 1 involves surmounting a barrier Δ, of distance 2, 2Δ, etc., so that barriers linearly grow with distance m, i.e., $\Delta_m=m\Delta$ and $a_k=R^k$, where $R\equiv e^{-\Delta/T}$. This is similar to the case studied in Ref. 6 and also corresponds to model (a),(d) of PSAA. Using Eqs. (4) and (6) and normalization of probabilities, we find

$$P_0(t)=2^{-n}+\tfrac{1}{2}\exp[R^{n+1}t/(1-R)]\sum_{m=0}^{n-1}\exp\{-m\ln 2-[(2-R)/(1-R)]R^{m+1}t\}. \tag{9}$$

This formula is exact. We now take the limit $n\to\infty$, recalling that $R<1$, and convert the sum to an integral. The integral can be solved exactly in closed form, and we find

$$P_0(t)=\frac{T\ln 2}{\Delta}\left[R\left(\frac{2-R}{1-R}\right)\right]^{-T\ln 2/\Delta}t^{-T\ln 2/\Delta}\gamma\left[\frac{T\ln 2}{\Delta},tR\left(\frac{2-R}{1-R}\right)\right], \tag{10}$$

where $\gamma(a,b)$ is an incomplete gamma function. As $t\to\infty$,

$$P_0(t)\sim t^{-T\ln 2/\Delta}-O(e^{-t}/t), \tag{11}$$

and we find a temperature-dependent power law with an exponent which vanishes as $T\to 0$ and diverges as $T\to\infty$. This high-temperature limit is to be expected because as $T\to\infty$ a hop to an infinitely distant site is as probable as a local hop.

While we can write down an exact formula for $\langle d(t)\rangle$ for finite n, it is really the long-time behavior as $n\to\infty$ that interests us, and we will simply present that:

$$\lim_{t\to\infty}\langle d(t)\rangle\sim(T/\Delta)\ln t. \tag{12}$$

In fact, this result can be rederived by a very simple scaling argument: If we rescale the time by a factor R^{-1}, all sets of neighboring points on the top level become indistinguishable, and we are left with an effective lattice which is one level lower. This results in a shift of 1 in the ultrametric distance, which again leads exactly to Eq. (12).

With Eqs. (6) and (7) we can solve any reasonable case of interest. Rather than solve a variety of special cases, however, we make the following observations about the general behavior of the dynamics of simple ultrametrics (i.e., uniformly bifurcating or multifurcating trees).

(1) The dynamics [i.e., behavior of $P_0(t)$ and $\langle d(t)\rangle$] on any single space is *not* universal, as has sometimes been claimed. The long-time dynamics depends crucially on the rate of increase of barriers Δ_m with distance m. The slower the rate of increase of Δ_m, the faster the relaxation at long times.

(2) There is a minimal rate of growth of Δ_m with m below which the random walk becomes unstable; that is, if Δ_m does not increase fast enough with m, then $P_0(t)=0$ and $\langle d(t)\rangle=\infty$ (in the $n\to\infty$ limit) for all $t>0$. This is easy to understand; suppose, for example, that all barriers are equal (as in the $T\to\infty$ case, for example). Then any jump of any distance is equally probable, and since the number of sites a distance m from any point diverges as 2^{m-1}, the theory cannot then be convergent.

We find that the slowest rate of growth of Δ_m with m that leads to a stable random walk is the case $\Delta_m=\Delta\ln m$ (for large m) with the requirement that $\Delta>T$; here $\Delta=T$ is the boundary between stability and instability. The long-time behavior of $P_0(t)$ in this case is the famous Kohlrausch law,

$$P_0(t) \sim e^{-t^\beta}, \tag{13}$$

with $\beta = T/\Delta$. When $T \rightarrow \Delta^-$, we approach Debye behavior, and for $T > \Delta$, the model becomes unstable.

The picture presented here is qualitatively independent of whether there are twofold, threefold, or p-fold multifurcations at each point. However, ultrametric space need not uniformly multifurcate, and we have not studied the behavior of such random ultrametric models.

In summary, we have found an exact solution for the problem of dynamics on an ultrametric space. Long-time behavior is dependent on both the structure of the space and the increase of barriers with distance. Linear increase of barriers yields a temperature-dependent algebraic decay of the autocorrelation function. The marginal case corresponds to logarithmic increase of barriers which yields a Kohlrausch law with temperature-dependent β. Models with barriers increasing with distance slower than logarithmically are unstable. The picture discussed here differs from that of the mean-field spin-glass: The transition probability from state a to b equals that from b to a for all states (implying that all states are degenerate); ultrametric distance is defined in terms of (finite) barriers, and the space uniformly bifurcates at all levels. However, it is intriguing to note that the recent analysis[10] of the dynamics of a *short*-ranged Ising spin-glass in three dimensions shows that, above T_g, a power-law decay is observed for dynamic processes taking place on length scales within the coherence length, and the decay crosses over at longer times to the faster Kohlrausch law. Both the power-law and Kohlrausch exponents were found to depend linearly on T in the critical regime.

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