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Chaos-Order-Chaos Transitions in a Two-Dimensional Hamiltonian System

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We present a new result which shows the transitions from chaos to order and again to chaos as the coupling parameter between two nonlinearly coupled oscillators of a Hamiltonian system is varied continuously from $-\infty$ to $+\infty$. By exploiting the symmetry of the system, we show that there is no general correspondence between the classical chaotic motion and the Gaussian-orthogonal-ensemble distributions of the energy-level fluctuations of the corresponding quantum system.

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The problem of understanding the onset of chaotic behavior in nonlinear Hamiltonian systems, where the chaotic motion is generated by the dynamics itself and not by external perturbation, is of considerable importance and has applications to a number of areas in science and engineering.[1] As in many nonlinear dissipative systems, Hamiltonian systems with at least two degrees of freedom are known to exhibit a transition from regular to chaotic motion as the energy of the system is increased, as shown, for example, by the well-studied two-dimensional Henon and Heiles system.[2] However, no systematic study seems to have been undertaken which carefully examines the important role played by the coupling parameters in a Hamiltonian system, without which the system would decouple into one-dimensional systems which can never exhibit chaotic motion. One of the novel results which we will report in this Letter is that as a function of the coupling parameter, the behavior of the two-dimensional Hamiltonian system which we studied is not simply divided by a single critical coupling parameter into a regular and chaotic region, but is divided by two critical coupling parameters into two chaotic regions separated by a regular region. Another novel feature of this Hamiltonian system is that its symmetry properties allow us to make a definite conclusion regarding the statistics of the energy-level fluctuations of its corresponding quantum mechanical system. Our conclusion from this symmetry consideration disproves arguments put forward recently[3] in which it was asserted that the quantum correspondence to the classical chaotic region could be associated with the energy-level fluctuations, obeying the Gaussian-orthogonal-ensemble (GOE) distribution, obtained from the random matrix theory.

The Hamiltonian of the two-dimensional nonlinear system which we have chosen for our study is

$$H=\tfrac{1}{2}(p_x^2+p_y^2+x^2+y^2)+\lambda(x^4+2Cx^2y^2+y^4), \quad (1)$$

where the x, y, p_x, p_y represent the displacements and momenta of the two oscillators, C represents the coupling parameter, and λ is a scaling parameter. The classical dynamical equations of motion are

$$\ddot{x}+x+4\lambda(x^3+Cxy^2)=0, \quad (2a)$$

$$\ddot{y}+y+4\lambda(y^3+Cx^2y)=0. \quad (2b)$$

The corresponding quantum system is one whose Hamiltonian operator is

$$\hat{H}=\frac{1}{2}\left(-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}+x^2+y^2\right)+\lambda(x^4+2Cx^2y^2+y^4), \quad (3)$$

which has arisen in problems associated with molecular dynamics,[4] bistable lasers,[5] and other topics.

The system first distinguishes itself by the following property: It is a classically integrable system (quantum mechanically separated system) at three particular values of the coupling parameter, $C=0$, 1, and 3. The first, $C=0$, clearly decouples the two oscillators. The second, $C=1$, reduces the system into a circularly symmetrical one and thus decouples the radial and the angular parts in the polar coordinates. The significance of the third, $C=3$, is less apparent but can be proved by making the following coordinate transformation. Let

$$x'=2^{-1/2}(x+y), \quad (4a)$$

$$y'=2^{-1/2}(x-y); \quad (4b)$$

then Eqs. (2) and (3) become

$$\ddot{x}'+x'+4\lambda'(x'^3+C'x'y'^2)=0, \quad (5a)$$

$$\ddot{y}'+y'+4\lambda'(y'^3+C'x'^2y')=0, \quad (5b)$$

and

$$\hat{H}=\frac{1}{2}\left(-\frac{\partial^2}{\partial x'^2}-\frac{\partial^2}{\partial y'^2}+x'^2+y'^2\right)+\lambda'(x'^4+2C'x'^2y'^2+y'^4), \quad (6)$$

where

$$\lambda'=\tfrac{1}{2}(1+C)\lambda, \quad (7)$$

$$C'=(3-C)/(1+C). \quad (8)$$

It follows from Eqs. (5)–(8) that $C=3$ corresponds to a system of two uncoupled oscillators ($C'=0$) with $\lambda'=2\lambda$. It is seen that the regime characterized by $C>1$ in the (x,y) coordinate system can be mapped into the regime characterized by $-1<C'\leq 1$ in the (x',y') coordinate system.

Thus, for three particular values of C, namely 0, 1, and 3, the behavior of the classical motion is always regular and never chaotic, no matter what the initial energy of the system is. This is verified by our numerical studies. Quantum mechanically, the energy levels at these particular values of C can be characterized by the quantum numbers (n_x,n_y) for the uncoupled oscillators, and by the quantum numbers (n,l) for the rotors, and the degeneracies of the energy levels are well known.

The questions are as follows: What about other values of C? Would they all give chaotic motion for any given initial energy? Is there any characteristic level fluctuation in the corresponding quantum system?

We have studied the problem numerically by examining, for each value of C, the following pictures of the motion: (i) the trajectory plot of $y(t)$ vs $x(t)$; (ii) the phase-space plot of $\dot{x}(t)$ vs $x(t)$; (iii) the power-spectrum plot of $x(t)$; and (iv) the largest Lyapunov exponent.

We have used all of the above pictures so as to be able to identify consistently the chaotic behavior from the regular behavior. For the initial conditions $x(0)=5$, $y(0)=10$, and $\dot{x}(0)=\dot{y}(0)=0$, the set of trajectory plots (i) for different values of C is shown in Fig. 1(a); the set of phase-space plots (ii) corresponding to $y(0)=0$ is given in of Fig. 1(b); and the power-spectrum plot of $x(t)$, where the horizontal coordinate gives the frequency and the vertical coordinate gives the logarithm of the absolute square of the Fourier transform of $x(t)$, is given in Fig. 1(c). Together with the Lyapunov exponents, these plots lead us to the clear conclusion that for the given initial conditions, the behavior of the motion is chaotic for $C\leq -0.21$, regular for $-0.21<C<5.2$, and again chaotic for $C\geq 5.2$.

Thus the motion turns from one of chaotic to regular and again to chaotic as C is continuously varied from $-\infty$ to $+\infty$. In the regions around the special values of $C=0,1,3$ for which the system is integrable, the motion remains regular even though the equations of motion are nonintegrable. Equations (4)–(8), which showed that the region $C>1$ can be mapped into the region $-1<C'<1$, can be used to deduce certain qualititave features of the chaotic and regular regimes under different initial conditions. For example, if it is known that under the initial conditions $x(0)=x_0$, $y(0)=y_0$, $\dot{x}(0)=\dot{y}(0)=0$ the motion is chaotic in the region $C_1\leq C\leq C_2$, then the motion must be chaotic in the region $(3-C_2/(1+C_2)\leq C\leq(3-C_1)/(1+C_1)$, given the initial conditions $x(0)=2^{-1/2}(x_0+y_0)$, $y(0)=2^{-1/2}(x_0-y_0)$, $\dot{x}(0)=\dot{y}(0)=0$. This was confirmed by our numerical results which showed the internal consistency of our conclusions. These transformation properties are useful when we examine the changes in the boundaries of the chaotic and regular regimes as the initial conditions are changed. We will report the details of

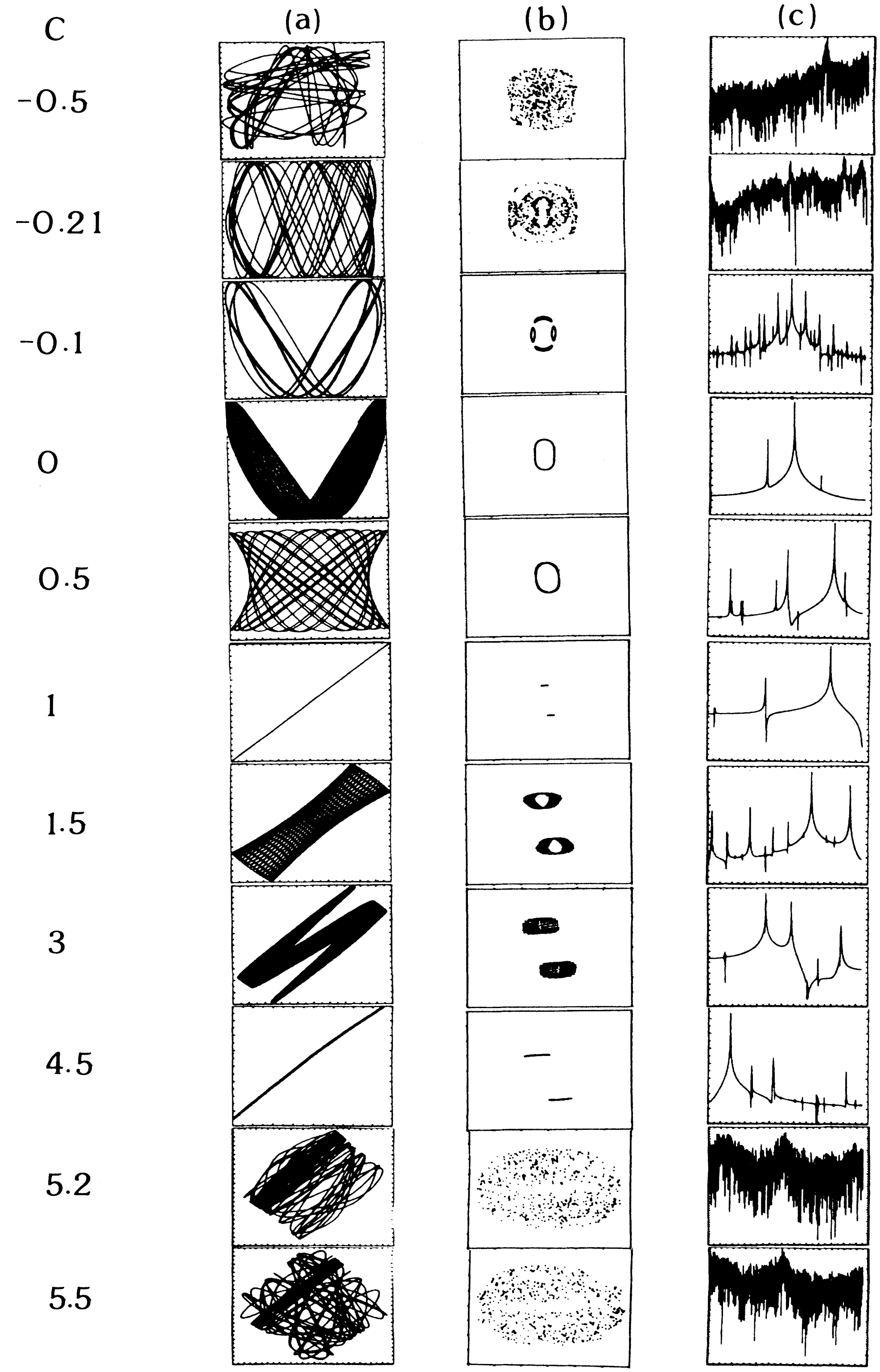

FIG. 1. (a) The trajectory plot of $y(t)$ (vertical axis) vs $x(t)$ (horizontal axis). (b) The phase-space plot of $x(t)$ (vertical axis) vs $\dot{x}(t)$ (horizontal axis) with $y(t)=0$; a total of 500 points have been included in each panel. (c) The power-spectrum plot of $x(t)$. The corresponding values of C are listed in the first column.

these studies elsewhere.

The second question concerns the quantum correspondence to these classical regular and chaotic regions. It is not difficult to show that our Hamiltonian operator (3), in the region $-1 < C < \infty$ in which the oscillators are bounded, has the following symmetry property: The energy levels characterized by the quantum numbers $(n_x, n_y) = (a,b)$ and (b,a), which are degenerate when $C=0$, split into two different energy levels when C differs from 0 if a and b are both odd or both even integers, but remain degenerate if one of the quantum numbers is an odd and the other is an even integer. It follows that for any value of the coupling parameter C, at least half of the energy levels of (3) are doubly degenerate. Hence we expect that the nearest-level distance distribution $P(x)$ has a sharp peak at $x=0$, where x is the energy difference of the nearest-neighbor levels. Our numerical result as shown in Fig. 2 confirms this conclusion. This result directly disproves the recent suggestions of Bohigas *et al.*[3] that in the classical chaotic region, the corresponding quantum energy-level fluctuations obey the Gaussian-orthogonal-ensemble distribution (GOE).[6] For the GOE distribution the level fluctuations $P(x)$ should be zero at $x=0$ because of the level-repulsion effect of random-matrix theory. Even if we put aside the question of dependence on the initial conditions in the classical motion, our exact symmetry argument for the degenerate energy levels of our system and our consistent identification of the corresponding classical chaotic motion in certain regions of C values by several different numerical methods clearly show that there is no general correspondence between the GOE distribution of the quantum level fluctuations and the classical chaotic motion.

In summary, our studies of the nonlinear Hamiltonian system (1) have revealed a novel feature of chaotic $\rightarrow$ regular $\rightarrow$ chaotic transitions as the coupling parameter is continuously varied from $-\infty$ to $+\infty$. When the initial energy is increased, more choatic regions are found to appear. These results, together with an exact symmetry consideration for the energy levels of the corresponding quantum system, also show that there is no direct correspondence between the GOE distribution of the energy-level fluctuations and the classical chaotic motion.

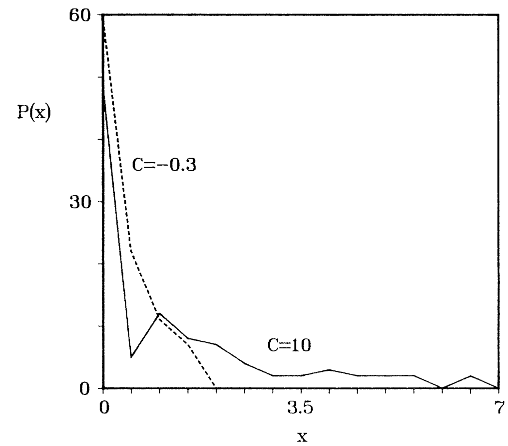

FIG. 2. The nearest-neighbor spacing distribution of the quantum level fluctuation for the eigenvalues of the Hamiltonian given by Eq. (1) with $C=-0.3$ and $C=10$. Only the lowest 100 levels are taken into account.

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[1]See, e.g., A. J. Lichtenberg and M. A. Lieberman, *Regular and Stochastic Motion* (Springer-Verlag, New York, 1983), and references therein.

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