Fractal Basin Boundaries and Homoclinic Orbits for Periodic Motion in a Two-Well Potential

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(Received 2 July 1985)

A fractal-looking basin boundary for forced periodic motions of a particle in a two-well potential is observed in numerical simulation. The fractal structure seems to be correlated with the appearance of homoclinic orbits in the Poincaré map as calculated by Holmes using the method of Melnikov. Below this critical forcing amplitude the basin boundary appears to be smooth and nonfractal. This example raises questions about predictability in nonchaotic dynamics of nonlinear systems.

PACS numbers: 03.20.+i, 05.40.+j, 46.30.−i

In a series of papers and reports, Yorke and coworkers[1–3] presented numerical evidence for fractal boundaries between basins of attraction for nonchaotic attractors. In their study they looked at two-dimensional maps with multiple nonchaotic attractors. In a recent lecture, Yorke suggested that equations representing flows of dynamical systems with multiple nonchaotic attractors, i.e., equilibrium points, limit cycles, periodic orbits, etc., might possess fractal boundaries between the two or more basins of attraction. This would imply that for small uncertainty in the initial conditions near this boundary absolute predictability might be impossible even if a solution is proved to exist and is unique. In a recent Letter[4] fractal basin boundaries have in fact been found for the forced pendulum.

In this Letter we have applied these ideas to the problem of forced motions of a particle in a two-well potential, and relate the appearance of fractal basin boundaries to the appearance of homoclinic orbits in the Poincaré map as examined by Holmes[5] in an earlier study. The two-well potential describes the motion of a buckled elastic beam or an electron in a plasma.[5–8] The governing equation under study is

$$\ddot{x}+\gamma\dot{x}-\tfrac{1}{2}x(1-x^2)=f\cos\omega t. \quad (1)$$

The importance of this model is that the chaotic and nonchaotic dynamics have been analyzed in great detail by Holmes[5] as well as by numerical simulation and by analog computer. The results of this work have been verified in experiments by Moon[6] for a buckled elastic beam. Because of the close agreement between theory, experiment, and numerical simulation, we have confidence that the numerical results presented in this paper reflect the actual properties of the dynamical system (1) and the physical systems it claims to model.

In his theoretical study of (1), Holmes used the method of Melnikov to derive a necessary criterion for chaotic motion based on the existence of homoclinic orbits in the Poincaré map when $f>f_c$, where

$$f_c=(\gamma\sqrt{2}/3\pi\omega)\cosh(\pi\omega/\sqrt{2}). \quad (2)$$

This criterion gives the condition for the intersection of stable and unstable manifolds associated with the saddle point of the Poincaré map ($\omega t=2\pi n$; n is an integer).

It is the thesis of this paper that this criterion is a necessary condition for the appearance of fractal basin boundaries between two periodic attractors and that unpredictability in the presence of uncertainties in initial conditions may be a property of the two-well potential even when the attractors are not chaotic.

Experiments on the forced motion of a buckled beam have shown that below some critical $f\equiv f_c^*$ ($f_c^*>f_c$) the motion is periodic and above f_c^* the motion may be chaotic. Fractal properties of the boundary between periodic and chaotic motions for this equation have recently been studied experimentally,[7] and the fractal dimension of the Duffing-Holmes attractor has recently been calculated by the authors.[8]

For the frequency $\omega=0.833$ and damping $\gamma=0.15$ this equation shows chaotic behavior for $f\geqslant 0.159$ when a fourth-order Runge-Kutta numerical integration is used while the critical Holmes value from (2) is $f_c=0.088$. In the present study we looked at the *nonchaotic regime* $0.05<f<0.1$. For long times, only periodic motions are possible, an orbit about the right or the left equilibrium position $x=\pm 1$, $\dot{x}=0$. We then explored the initial-condition space (x_0, $\dot{x}_0=v_0$) to find the basin of attraction of these two attractors.

The criterion used to determine the long-term state of the orbit was first to ignore the initial transient, equivalent to five driving periods, and then wait until the trajectory has made five orbits about either the left or the right equilibrium position $x=\pm 1$ by looking at the long-time average of $x(t)$. If the orbit went to the right attractor we plotted a symbol; if it went to the left attractor we left a blank. In this way the edge of the dark symbols represents the basic boundary.

The numerical results are plotted in Figs. 1–5. For low values of f ($\omega=0.833$), $f\sim 0.05$, the boundary looks smooth as shown in Fig. 1. However, for $f=0.1$, which is greater than the Holmes critical value of 0.088, the coarse-scale boundary shows some

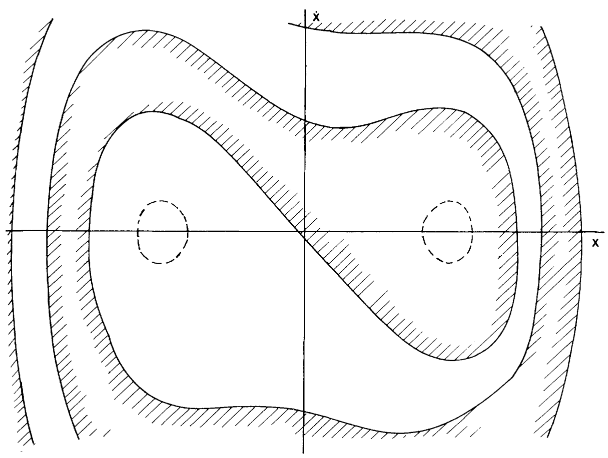

FIG. 1. Smooth basin boundary (solid lines) for periodic motion about equilibrium points $x = \pm 1$, $\dot{x} = 0$ (dotted line) for forcing amplitude $f = 0.05$ and frequency $\omega = 0.833$ (damping $\gamma = 0.15$).

fingers or whiskers indicating possible fractal behavior as shown in Fig. 2 for 400×400 initial conditions. To confirm this we performed a sequence of successive enlargements of smaller and smaller regions of phase space near the suspected fractal boundary as shown in Fig. 3. Each enlargement consisted of 10^4 initial conditions (100×100) and each showed finer and finer structure indicating a possible fractal boundary. A composite photograph with 400×400 initial conditions is shown in Fig. 3.

All of the numerical results were obtained with use of a VAX 750 computer. Most of the data were run with a Runge-Kutta solver with a step size of 0.25. However, the data in Fig. 2 were also run with a 0.1 step size and the results were almost identical. Further, a dozen or more individual points from Fig. 2 were selected at random near the fractal-looking boundary and run for a long time to make sure that the criterion for left or right periodic attractor was operating correctly. In all cases the long-time orbits were periodic. Most were period-one orbits, but a few were period three as judged by use of Poincaré maps.

In a paper on the forced Duffing equation (1), Holmes[5] showed that the chaotic motions were preceded by the appearance of an infinite set of homoclinic orbits in the Poincaré map of the periodically forced system for a critical value of the forcing amplitude f. These orbits occurred when the unstable and stable manifolds, emanating from the saddle point at the origin, intersected.

For low damping, however, $\gamma < 0.2$, the first author has shown[6] in experiments on vibrations of a buckled elastic beam that the condition (2) was only a necessary one for chaotic vibrations of a buckled beam (i.e., $f_c < f_c^*$). Thus there lies a region in the parameter space (f, ω, γ) where homoclinic orbits exist but chaos is not likely (i.e., for $f_c < f < f_c^*$). We conjecture that Holmes's criterion (2) may give the critical value of "f" for a fractal basin boundary between two nonchaotic periodic attractors about the left or right equilibrium points.

First, this conjecture is supported by the data in Fig. 4. Here the results of Runge-Kutta simulation are compared with the Holmes criterion (2). The circles indicate that smooth, nonfractal looking boundaries were obtained similar to Fig. 1. The star data points indicate the appearance of a nonsmooth boundary. Because of the costs and computer time involved only one frequency ($\omega = 0.833$) was explored on a finer

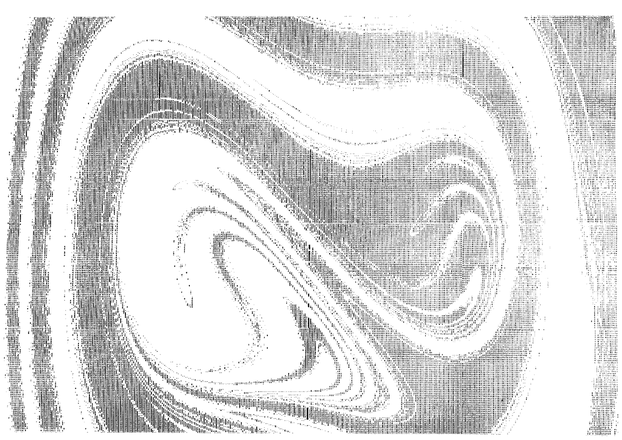

FIG. 2. Fractal-looking basin boundary for forcing amplitude $f = 0.1$ ($\omega = 0.833$, $\gamma = 0.15$) calculated from 160×10^3 initial conditions in the domain $-2.4 \leq x_0 \leq 2.4$, $-1.2 \leq v_0 \leq 1.2$.

FIG. 3. Composite photograph of finer scale enlargement of Fig. 2 basin boundary for 160×10^3 initial conditions, $f = 0.1$, $\omega = 0.833$, $\gamma = 0.15$, $-0.375 \leq x_0 \leq -0.275$, $0.045 \leq v_0 \leq 0.075$.

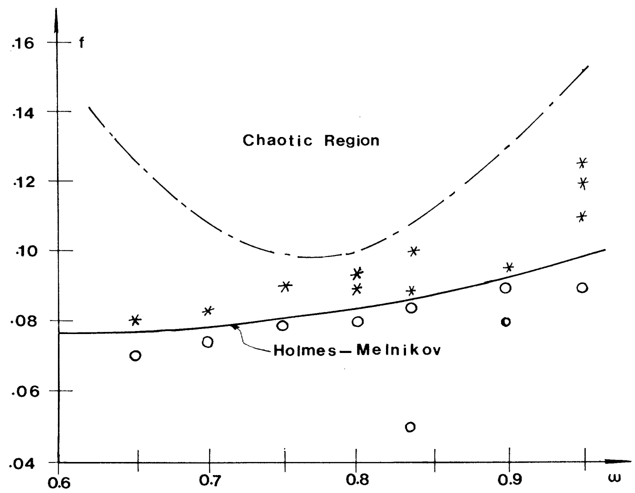

FIG. 4. Comparison of the Holmes criterion for homoclinic orbits with numerical evidence for smooth and nonsmooth basic boundaries. The lower bound of the chaotic region is shown by the dashed curve.

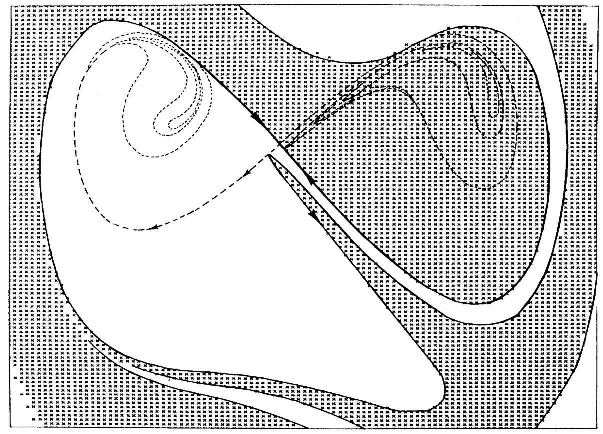

FIG. 5. Stable and unstable manifolds of the Poincaré map superimposed on the basin boundary for forcing amplitude at the Holmes critical value ($f=0.0856$, $\omega=0.8$, $\gamma=0.15$).

scale to ascertain the self-similar, fractal nature of the boundary. The dashed line represents a lower bound on the chaos criterion.

Second, there is numerical evidence that the appearance of fractal-looking structure in the basin boundary is coincident with the intersection of stable and unstable manifolds of the Poincaré map as shown in Fig. 5. When $f < f_c$, one can argue that the stable manifold of the Poincaré map and the basin boundary are coincident. [This conclusion was also found for an approximate two-dimensional cubic map associated with (1) by Yamaguchi and Mishima.[9]] Using Holmes's results we show in Fig. 5 the saddle point of the Poincaré map calculated from (1). It is evident from Fig. 5 that at the Holmes criterion, the stable manifold develops a fold or finger which touches the unstable manifold shown as a dotted curve. Thus it appears that the criterion for homoclinic orbits in the forced two-well potential problem is coincident with the change from a smooth to an irregular and perhaps fractal basin boundary.

Yorke and co-workers have shown that the fraction ϕ of uncertain initial conditions in the phase space as a function of the radius of the sphere of uncertainty ϵ in initial conditions has the following relation:

$$\phi \sim \epsilon^{D-d}, \tag{3}$$

where D is the dimension of the phase space and d is the fractal dimension of the basin boundary. For example, for $d=1.5$, an uncertainty in initial conditions of $\epsilon=0.01$ (compared to order one) yields an uncertainty fraction of 10%. For $\epsilon=0.05$, $\phi=22\%$.

Fractal properties of chaotic motions have been of course the subject of great interest in the past decade. Such behavior is associated with a sensitivity to initial conditions and a loss of information about the motion as time proceeds. The importance of the conjectures of Yorke and co-workers is that a larger class of nonlinear phenomena may suffer from inherent unpredictability than was previously thought. This includes transient and periodic as well as chaotic problems. This discovery is ironic in the age of the supercomputer in which numerical simulation through finite-element, finite-difference, and other CAD/CAM software promises to increase our analysis and prediction capability of the physical world.

A sensitivity of numerical simulation predictions of nonlinear phenomena has been known anecdotally in the numerical prediction industry. Recently a study by Symonds[10] has appeared in the literature which is somewhat related to the problem in this paper. There he tried to predict the end-state behavior of the transient excitation of an elastic-plastic beam arch. The end states involve periodic oscillations about two possible buckled positions of the beam arch. Thirteen different investigators ran the same problem with different numerical codes and obtained different answers. Such inability to obtain consistent results from numerical codes may be the consequence of fractal basic boundaries in either the initial-condition space or the parameter space for the nonlinear problem.

This work was supported in part by a grant from the U.S. Air Force Office of Scientific Research, Mathematical Sciences and Aerospace Division.

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