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N-Dependent Fractional Statistics of N Vortices

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A system of N identical point vortices on a plane, e.g., a superfluid-^{4}He thin film, is studied with use of elementary quantum mechanics. We find the surprising result that the system obeys fractional or θ statistics *which depends on N*, where $\theta(N) = \pi/N$ or $\pi/N + \pi$. This result arises from the angular momentum contained in the zero-point motion of the vortices.

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Fractional topological quantum numbers have recently been emerging as a common feature of condensed-matter physics in lower dimensions, and lower-dimensional relativistic field theories. Specifically, solitons with fractional charge and fermion number,[1] and with fractional spin and statistics,[2] have appeared in a number of contexts. A notable example is the $\frac{1}{3}e$ quasiparticle in the quantum Hall fluid.[3] In a previous paper,[4] we argued that the vortex in a thin film of superfluid, which is a two-dimensional soliton with a topologically nontrivial hole at its center, possesses fractional, or θ, statistics. This we did by studying the quantum dynamics of two identical vortices. We concluded that a single vortex in this system obeys quarter-fractional statistics (i.e., $\theta = \pi/2$ or $3\pi/2$). Hence, it is neither a boson ($\theta = 0$) nor a fermion ($\theta = \pi$).

Here we generalize this previous result to the case of N identical point vortices on a plane. These N vortices carry N holes, with global consequences. The surprising outcome of this present study is that θ for this system depends on the number of vortices: We find that $\theta(N) = \pi/N$ or $\pi/N + \pi$. It has been recently pointed out that the braid group *allows* representations where θ depends on the total number of particles. Ringwood and Woodward,[5] in the specific case of 't Hooft–Polyakov monopoles, have suggested that the set of *allowed* values of θ depends on the number of monopoles. More generally, Thouless and Wu[6] have argued that the braid group acting on particles on a sphere leads to a number dependence of the set of *possible* θ values. However, in previous models, e.g., anyons, θ was assumed to be independent of the number of particles.[7] This then, is the first specific dynamical system for which N-dependent statistics is suggested as being *necessary*.

A real physical system of vortices, e.g., in superfluid-^{4}He thin films, arises nontrivially from the underlying many-body problem. Here, for want of a satisfactory many-body theory in general, we start not from the microscopic picture, but from a macroscopic effective Hamiltonian for point vortices. A justification for this lies in the fact that the Kosterlitz-Thouless theory, which starts with the same Hamiltonian, has been experimentally verified.[8] The key assumptions which we share in common with the

Kosterlitz-Thouless theory are (I) that the vortices are *pointlike,* (II) that the vortices move according to a Hamiltonian that has no kinetic term, and (III) that the thin film constitutes a *two-dimensional, incompressible* fluid. However, we go on to make the further assumptions (IV) that *canonical quantization* of vortex motion can be carried out, and (V) that the resulting angular momentum operator [see Eq. (6) below] has no arbitrary additive constant. We would like now to discuss these assumptions. Concerning (I), the *pointlike* vortex is assumed to be a soliton carrying only one hole, not two or more nearby holes, at its core. It has been shown that such solitons, with separations large compared with their core sizes, can be approximated as point particles obeying Kirchhoff's equations [Eqs. (1)-(4) below].[9] Concerning (II), the lack of a kinetic term in the Hamiltonian can be interpreted as the vortices being *massless.*[10] Physically, each vortex is carried along with the local velocity field resulting from all the other vortices in the system. This assumption has experimental support.[11] Concerning (III), the *two-dimensionality* and *incompressibility* of a near-monolayer film arise from strong van der Waals forces which bind ^{4}He atoms so tightly to the substrate that all motions perpendicular to the substrate, e.g., ripplons, are quantized, and hence frozen out at low temperatures, to some level of approximation. Thus, the system is effectively two dimensional. Similarly, vortex motions are effectively decoupled from all frozen compressible degrees of freedom, e.g., phonons. Concerning (IV), we argue that *canonical quantization* is necessary, because such a procedure, which has been used previously for vortex dynamics,[12] avoids a violation of the uncertainty principle by *massless* vortices. Furthermore, the requirement that the angular momentum of the superfluid be quantized is equivalent to quantization of vortex motion.[4] Concerning (V), we argue that the absence of an arbitrary constant in Eq. (6) arises from a description of the system in an inertial frame.

The classical equations of motion obeyed by N identical point vortices of counterclockwise circulation $\kappa > 0$ centered at (x_i, y_i), $i = 1, 2, \ldots, N$, are

$$\kappa\, dx_i/dt = +\partial H_*/\partial y_i, \tag{1}$$

$$\kappa\, dy_i/dt = -\partial H_*/\partial x_i, \tag{2}$$

where, after subtraction of the self-energy of individual vortices, and definition of a as a scale factor,

$$H_* = -(\kappa^2/4\pi) \sum_{i<j} \ln(r_{ij}^2/a^2), \tag{3}$$

$$r_{ij}^2 = (x_i - x_j)^2 + (y_i - y_j)^2. \tag{4}$$

For a film of density ρ and thickness δ, the energy of the system is $H = (\rho\delta)H_*$. By inspection, these equations are Hamiltonian in form. One classical solution to these equations consists of N identical, equally spaced vortices on a circle, where each vortex is carried along by the resultant velocity field of all the other vortices symmetrically. This symmetry implies that $N-1$ consecutive transpositions of a given vortex with its nearest neighbor to the right on the circle is equivalent to a counterclockwise rotation of the entire circle around its center by an angle of $2\pi/N$. We seek a quantum generalization of this solution.

To quantize these equations of motion, we employ the standard method of canonical quantization. By inspection of Eqs. (1) and (2), we see that the canonically conjugate variables are x_i and y_i. Hence[12]

$$[x_i, y_j] = iC\delta_{ij}, \tag{5}$$

where $C = \hbar/\kappa\rho\delta$.[13] Defining the operator

$$L_N = -\frac{\hbar}{2C} \sum_{i=1}^{N} (x_i^2 + y_i^2), \tag{6}$$

one can show from Eq. (5) that

$$[L_N, x_i] = +i\hbar y_i, \tag{7}$$

$$[L_N, y_i] = -i\hbar x_i. \tag{8}$$

Hence, L_N is the generator of infinitesimal rotations, i.e., the angular momentum of the N-vortex system, in agreement with the classical expression.[14] To find the eigenfunctions of L_N, we write

$$y_i = (C/i)\partial/\partial x_i, \tag{9}$$

so that L_N is formally identical to the sum of N one-dimensional uncoupled simple harmonic-oscillator Hamiltonians. The eigenvalue spectrum of L_N consists of multiples of $\frac{1}{2}\hbar$, because of zero-point motion. By inspection, a totally symmetric state of the system in x coordinates is

$$\Psi_0^{(+)}(x_1, x_2, \ldots, x_N) \propto \exp\left(-\frac{1}{2}\alpha^2 \sum_{i=1}^{N} x_i^2\right), \tag{10}$$

where $\alpha^2 = |C|^{-1}$. It is also possible to construct a totally antisymmetric wave function in the usual way, with use of the Slater determinant,

$$\Psi_0^{(-)}(x_1, x_2, \ldots, x_N) \propto \begin{vmatrix} H_0(\alpha x_1)e^{-\alpha^2 x_1^2/2} & \cdots & H_0(\alpha x_N)e^{-\alpha^2 x_N^2/2} \\ \vdots & & \vdots \\ H_{N-1}(\alpha x_1)e^{-\alpha^2 x_1^2/2} & \cdots & H_{N-1}(\alpha x_N)e^{-\alpha^2 x_N^2/2} \end{vmatrix}, \tag{11}$$

where $H_n(\alpha x_i)$ are the Hermite polynomials. These states give the smallest angular momentum eigenvalues for the symmetric and antisymmetric cases, respectively:

$$L_N \Psi_0^{(+)} = -\tfrac{1}{2} N\hbar \Psi_0^{(+)}, \tag{12}$$

$$L_N \Psi_0^{(-)} = -\tfrac{1}{2} N^2\hbar \Psi_0^{(-)}. \tag{13}$$

Excited states, symmetric or antisymmetric, are easy to construct from these states by use of raising operations. This procedure provides a *complete* and *unique* solution to the angular momentum eigenvalue problem. Whether the vortices should be in a symmetric or antisymmetric state is not determined from the theory. Usually an antisymmetric state indicates that it is formed by fermions, and a symmetric state that it is formed by bosons, but this is not so here. This is immediately obvious from the fact that the angular momentum for the symmetric state is *half-fractional* for N odd, which can never occur for bosons. In general, in order to find the phase factor upon interchange, $\exp(-i\theta)$, it would seem that one could interchange vortices i and j by exchanging x_i with x_j, *and then* y_i with y_j. But here one cannot specify both x_i and y_i since $[x_i,y_i]\neq 0$. Hence, symmetry or antisymmetry in the x coordinates alone does not imply Bose or Fermi statistics.

The angular momentum itself is the key to determining θ. It will prove convenient to work not with L_N directly, but rather with the operator l_N, in which the center-of-vorticity angular momentum is subtracted out:

$$l_N = L_N + \frac{N\hbar}{2C}(X^2+Y^2)$$
$$= -\frac{\hbar}{2NC}\sum_{i<j}[(x_i-x_j)^2+(y_i-y_j)^2], \tag{14}$$

where $X=\sum_{i=1}^N x_i/N$ and $Y=\sum_{i=1}^N y_i/N$ are the coordinates of the center of vorticity. It follows that

$$[X,Y]=iC/N, \tag{15}$$

$$[l_N,(x_i-x_j)] = +i\hbar(y_i-y_j), \tag{16}$$

$$[l_N,(y_i-y_j)] = -i\hbar(x_i-x_j), \tag{17}$$

$$[l_N,L_N]=[l_N,P_{ij}^{(x)}]=[L_N,P_{ij}^{(x)}]=0, \tag{18}$$

$$[H,l_N]=[H,L_N]=[H,P_{ij}^{(x)}]=0 \tag{19}$$

where $P_{ij}^{(x)}$ exchanges the x_i and x_j coordinates of $\Psi(x_1,x_2,\ldots,x_N)$. Since $(P_{ij}^{(x)})^2=1$, the one-dimensional unitary representations of the $P_{ij}^{(x)}$, the generators of the symmetric group S_N, are either $+1$ or -1, simultaneously for all (i,j) pairs. Hence, all angular momentum wave functions are either totally symmetric or totally antisymmetric. From Eqs. (12), (14), and (15), we get

$$l_N\Psi_0^{(+)} = -\tfrac{1}{2}(N-1)\hbar\Psi_0^{(+)}. \tag{20}$$

From Eqs. (16) and (17), we see that l_N is the generator of infinitesimal mutual rotations for all vortex pairs (i,j). From this it follows that the operator $\exp(i2\pi l_N/\hbar)$ transposes all $N(N-1)/2$ pairs of vortices *twice* in a counterclockwise sense. Each counterclockwise transposition results in a phase factor $\exp(-i\theta)$. Also, the operator $\exp(i2\pi l_N/\hbar)$ is equivalent to a counterclockwise rotation of the entire system around its center by an angle 2π.[15] Thus,

$$e^{i2\pi l_N/\hbar}\Psi_0^{(+)} = e^{-i\pi(N-1)}\Psi_0^{(+)}$$
$$= e^{-i\theta N(N-1)}\Psi_0^{(+)}. \tag{21}$$

This leads to the result

$$\theta=\pi/N. \tag{22}$$

Similarly from Eqs. (13), (14), and (15), we get

$$l_N\Psi_0^{(-)} = -\tfrac{1}{2}(N^2-1)\hbar\Psi_0^{(-)}. \tag{23}$$

Let us rewrite $\Psi_0^{(-)}$ as

$$\Psi_0^{(-)} \propto \prod_{i<j}(a_i^\dagger - a_j^\dagger)\Psi_0^{(+)}, \tag{24}$$

where $a_i^\dagger = |2C|^{-1/2}(x_i - iy_i)$ is the raising operator for the angular momentum of the ith vortex. In this form it is apparent that all pairs are excited identically. Repeating the argument used to determine θ for the $\Psi_0^{(+)}$ state, we find that

$$e^{i2\pi l_N/\hbar}\Psi_0^{(-)} = e^{-i\pi(N^2-1)}\Psi_0^{(-)}$$
$$= e^{-i\theta N(N-1)}\Psi_0^{(-)}. \tag{25}$$

This leads to the result

$$\theta=\pi/N+\pi. \tag{26}$$

In the limit $N\to\infty$ of Eqs. (22) and (26), we recover the usual Bose and Fermi statistics. Thus, the thermodynamic limit $N\to\infty$ is the usual one.

Here we have found θ by dividing the total phase change upon interchange of all pairs by the total number of pairs.[16] It would seem that a more direct way of defining θ would be to find the phase change resulting from an interchange of one pair along a counterclockwise loop, *keeping all other vortices outside.* However, it is impossible to construct an operator which will insure this last condition, since the vortices cannot be localized in configuration space, even in principle, since $[x_i,y_i]\neq 0$. In any case, the physically observable angular momentum, l_N, has unusual properties. The symmetric ground state, for instance, has an eigenvalue of l_N proportional to $N-1$ while for the N-independent θ statistics it would be proportional to

$N(N-1)/2$.[6,7]

In view of the N dependence of θ, one might ask: Is the relative motion of two nearby vortices affected by the addition of a third vortex far away? The answer is no, but there will be an additional superimposed rotation of all three vortices about their common center of vorticity. It is the zero-point angular momentum about this center which yields the N-dependent θ statistics.

All excited states have angular momentum values differing by $\hbar$ times an integer from the smallest angular momentum. The angular momentum spectrum is one sided corresponding to the fact that the classical orbits of the like-signed vortices is always of one sense only. Since all states are totally symmetric or antisymmetric, there are only two possible values of θ, which are given by Eqs. (22) and (26). Transitions between symmetric and antisymmetric states are forbidden. Hence, there should exist two distinct systems of N vortices ("ortho" and "para"), in each of which half of the states are missing as a result of statistics, like in a diatomic molecule with identical nuclei, e.g., H_2.

The Abrikosov vortex in type-II superconductors is essentially the same topological soliton as the vortex in superfluid ^4He.[17] Hence, these results should also apply to N Abrikosov vortices.

We believe that these results have a deeper topological significance. By inspection of Eqs. (1) and (2), one sees that the configuration space of the N vortices is *symplectic,* i.e., it has the geometrical structure of a *phase space* of $2N$ dimensions. This space is multiply connected. The unusual statistics arises from the zero-point motion of these vortices, which can be characterized by the Maslov topological index m.[18] Here $m=2N$. Also note that the specific logarithmic form of the Hamiltonian, Eq. (3), did not enter our analysis.

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[10] The vortex system is equivalent to a two-dimensional *massless* Coulomb gas in a constant magnetic field. A system of N charged particles with mass m moves on a plane pierced by a constant magnetic field $\mathbf{B}$, normal to it. The Lorentz force on the ith particle is then given by $m\ddot{\mathbf{r}}_i = e(\mathbf{E}_i + \dot{\mathbf{r}}_i \times \mathbf{B})$ where $\mathbf{E}_i = (e/2\pi)\sum_{j\neq i}\mathbf{r}_{ij}/|\mathbf{r}_{ij}|^2$ in two dimensions. Now, if we set $m=0$, Eqs. (1), (2), and (3) follow, with $\kappa = e/B$.

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[15] If $\xi_i = x_i - X$ and $\eta_i = y_i - Y$, then $[l_N, \xi_i] = +i\hbar\eta_i$ and $[l_N, \eta_i] = -i\hbar\xi_i$, so that l_N is also the generator of infinitesimal rotations *around the center of vorticity.*

[16] In excited states, $2\pi k$ where k is an integer must be added to the interchange phase change for some pairs to determine θ by this method, just as in ordinary Bose or Fermi gases.

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