Critical Behavior of Two-Dimensional Systems with Continuous Symmetries

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Conformal invariance allows a complete classification of critical theories in two-dimensional systems with continuous symmetries. We study the spin-$\frac{1}{2}$ chain by non-Abelian bosonization to show how it fits into this classification.

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Conformal invariance powerfully constrains two-dimensional critical phenomena.[1,2] It implies the existence of a local, conserved, traceless energy-momentum tensor, $T_{\mu\nu}(X)$, with $T_{00}=H$, the Hamiltonian density, and $T_{0,1}=P$, the momentum density. It is convenient to adopt light-cone coordinates, $x_\pm=x_0\pm x_1$; $T=(H-P)/2$, $\overline{T}=(H+P)/2$. The conservation equation $\partial^\mu T_{\mu\nu}=0$ implies $\partial_+T=\partial_-\overline{T}=0$ so that $T=T(x_-)$, $\overline{T}=\overline{T}(x_+)$. The most general possible commutation relations are

$$-i[T(x_-),T(x_-')]=\delta(x_--x_-')T'-2\delta'(x_--x_-')T+(c/24\pi)\delta'''(x_--x_-'), \tag{1}$$

and similarly for $\overline{T}$, with $[T,\overline{T}]=0$. (The central charge c will be the same for $\overline{T}$ if the system is invariant under parity, $x_-\leftrightarrow x_+$, which we will assume to be the case.) Equation (1) is the Virasoro algebra. If the central charge c is less than 1, the assumption of unitarity[2] yields a discrete set of possible values of c: $c=1-6/(n+1)(n+2)$ $[n=2,3,4,\ldots]$. For each of these values there is a discrete set of possible scaling dimensions (critical exponents). These theories describe critical and tricritical Potts models, which have only *discrete* symmetries.[2]

However, it is known that continuously variable critical points are possible for $c\geqslant 1$ and no restrictions have been placed on possible values of c. The purpose of this paper is to discuss the improvement in this situation for systems with *continuous* internal symmetries. Physically interesting systems with U(1) or (at special points) SU(2) symmetry include the six-vertex model, quantum spin chains, electron gases, and the Kondo problem. Other symmetries occur in the replica formulation of disordered systems, for example, SU($2n$) in the quantum Hall-effect localization theory.

The logical extension of the above assumption of conformal invariance to systems with continuous symmetries is the assumed existence of local, conserved currents $J_\mu=(\overline{J}+J,\overline{J}-J)$. Conformal invariance implies the existence of a well-defined scaling dimension for J_μ which must be 1 if J_μ is conserved. It then follows that J_μ has zero curl,[3] as can be seen by consideration of the (non-time-ordered) two-point function $\langle 0|J(x)J(x')|0\rangle$. Lorentz invariance demands that this have the form $f[(x-x')^2]/(x_--x_-')^2$. Scaling then requires f to be a constant. Thus the scalar field ∂_+J annihilates the vacuum and is therefore zero.[4] Similarly $\partial_-\overline{J}=0$ so that $J=J(x_-)$, $\overline{J}=\overline{J}(x_+)$. These two equations are equivalent to $\partial^\mu J_\mu=0$, $\partial^\mu[\epsilon_{\mu\nu}J^\nu]=0$. Thus the critical theory has always doubled conserved currents and *the symmetry of the model is of the form $G\otimes G$ at the critical point.* In many cases this means that a model with symmetry group G will have an enlarged symmetry group $G\otimes G$ at the critical point. This can occur if operators that break the additional (chiral) symmetry are irrelevant. This is fairly well known for $G=$ U(1). The Thirring (Luttinger) model is only critical when operators that break the chiral symmetry (independent conservation of left- and right-moving electron number) are absent (or irrelevant). We show below that this is also true for the isotropic spin chain [$G=$ SU(2)].

In the U(1) case (a single conserved, curl-free current, J_μ) it is convenient to solve the conservation equations in terms of a scalar field: $J_\mu\sim\epsilon_{\mu\nu}\partial^\nu\phi$, $\partial^2\phi=0$. The most general commutators consistent with Lorentz invariance and scaling are

$$[J(x_-),J(x_-')]=i\delta'(x_--x_-')/\sqrt{\pi} \tag{2}$$

(and similarly for $\overline{J}$ with $[J,\overline{J}]=0$; the constant can be fixed by rescaling the currents). These follow from the free scalar field theory: $L=\frac{1}{2}(\partial_\mu\phi)^2$.

Alternatively, we may observe that space-time translations of $J,\overline{J}$ are generated by the energy-momentum tensor $T=\frac{1}{4}\pi JJ$, $\overline{T}=\frac{1}{4}\pi\overline{J}\overline{J}$ [this follows from the commutators of Eq. (2)]. Thus, the full energy-momentum tensor must be the above one plus possible additional terms which commute with the current. If we assume irreducibility of the current (all operators that commute with it are functions only of the conserved charges) then these additional terms can be ignored. But this is the energy-momentum tensor of the free scalar field. Thus we conclude that any U(1)-invariant critical point should be described by a free scalar field. $T,\overline{T}$ obey the Virasoro algebra [Eq. (1)] with central charge $c=1$. A continuous set of possible scaling dimensions exists, corresponding to

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the operators $e^{i\beta\phi}$ (with dimension $\beta^2/4\pi$). This critical point describes a free massless fermion $L = \psi_L^\dagger i\partial_+\psi_L + \psi_R^\dagger i\partial_-\psi_R$. This can be shown, for example,[5] by observing that $T,\bar{T}$ are quadratic in the currents $J = i{:}\psi_L + \psi_L{:}$, $\bar{J} = i{:}\psi_R + \psi_R{:}$, which obey the commutators of Eq. (2). This amounts to a proof of the bosonization formula[6] $\bar{\psi}\gamma_\mu\psi = (1/\sqrt{\pi})\epsilon_{\mu\nu}\partial^\nu\phi$. The other standard formula,[6] $\psi_L^\dagger\psi_R \propto \exp[i(4\pi\phi)^{1/2}]$, can be obtained by observing that the commutators

$$[J(x),\psi_L+\psi_R(y)] = -[\bar{J}(x),\psi_L+\psi_R(y)] = \psi_L+\psi_R\delta(x-y)$$

are correctly reproduced.

In the non-Abelian case the most general possible commutators are

$$[J^a(x_-),J^b(x'_-)] = if^{abc}J^c(x_-)\delta(x_- - x'_-) + ik\delta^{ab}\delta'(x_- - x'_-)/2\pi, \tag{3}$$

and similarly for $\bar{J}$; $[J,\bar{J}]=0$. (Here f^{abc} are the structure constants of the group; $[T^a,T^b] = if^{abc}T^c$ with the conventional normalization $\mathrm{Tr}\,T^aT^b = \delta^{ab}/2$.)

Once we have normalized the currents to obtain the first term in the above equations, k then appears as an arbitrary parameter. However, it is known that this (Kac-Moody) algebra only has unitary representations for k a positive integer (see Witten[7] and references therein to the mathematical literature). Again, an energy-momentum tensor quadratic in currents can be constructed[8-10] which generates space-time translations of $J,\bar{J}$: $T = \pi J^aJ^a/4(c_v+k)$, $\bar{T} = \pi\bar{J}^a\bar{J}^a/4(c_v+k)$ (where c_v is the quadratic Casimir operator in the adjoint representation: $f^{abc}f^{abd} = c_v\delta^{cd}$). This tensor obeys the Virasoro algebra with $c = k\dim(G)/(c_v+k) \geqslant 1$. If we again assume irreducibility of the currents then this is the energy-momentum tensor on the full Hilbert space. The currents can again be represented[7] by scalar fields:

$$J^a = i\,\mathrm{Tr}g^{-1}\partial_+gT^a/2\pi, \quad \bar{J}^a = -i\,\mathrm{Tr}(\partial_-g)g^{-1}T^a/2\pi, \tag{4}$$

where g is a field in the group manifold [e.g., for SU(n) g is a special unitary $n\times n$ matrix]. The conservation equations become $\partial_-(g^{-1}\partial_+g) = 0$ $\{\Longrightarrow \partial_+[(\partial_-g)g^{-1}] = 0\}$. These equations, the current commutators, and the energy-momentum tensor can all be obtained[7] from the Wess-Zumino Lagrangean

$$S = k\,\mathrm{Tr}[\textstyle\int d^2x\,\partial_\mu g\,\partial^\mu g^{-1} + \frac{1}{3}\int d^3x\,\epsilon^{\mu\nu\lambda}g^{-1}\partial_\mu gg^{-1}\partial_\nu gg^{-1}\partial_\lambda g]/4\pi.$$

(The second term is defined by extending compactified two-dimensional space-time to the interior of a ball.[7]) The Kac-Moody algebra together with the quadratic energy-momentum tensor completely determine a discrete set of possible scaling dimensions.[8,9] A primary field[1] in the $(r,\bar{r})$ representation of $G\otimes G$ has scaling dimension $(c_r + c_{\bar{r}})/(c_v+k)$ where c_r is the quadratic Casimir operator in the r representation. *Thus if the symmetry group is non-Abelian there is a discrete set of possible critical theories given by the Wess-Zumino models.* (Even if the above irreducibility assumption is relaxed it is known that only a finite number of unitary representations of the Kac-Moody algebra exists for each integer k. See Ref. 6 and references therein to the mathematical literature.) For the case $G = \mathrm{SU}(n)$ and $k=1$, this critical point is again related to free fermions,[3,7,8] this time with Lagrangean $L = \psi_L^{\dagger i}\partial_-\psi_{Li} + \psi_R^{\dagger i}\partial_+\psi_{Ri}$ $(i=1,\ldots,n)$. The SU(n) and U(1) currents,

$$J^a = i{:}\psi_L^{\dagger i}(T^a)_i^j\psi_{Lj}{:}, \quad J = i{:}\psi_L^{\dagger i}\psi_{Li}{:} \tag{5}$$

(and similarly for $\bar{J}$), obey respectively the SU(n) Kac-Moody algebra with $k=1$ and the Abelian algebra of Eq. (2). The energy-momentum tensor can be written $T = \pi J^aJ^a/4(n+1) + \pi JJ/4n$ (and similarly for $\bar{T}$). Thus, the theory is equivalent to decoupled Wess-Zumino (g) and free-boson ϕ theories. The other bosonization rule,

$$\psi_L^{\dagger i}\psi_{Rj} \propto g_j^i\exp[i(4\pi/n\phi)^{1/2}], \tag{6}$$

can be obtained by considering the commutators of the currents with this object. Note that the scaling dimension of the field g can be read off from Eq. (6) by taking the difference between that for free fermions and that for the free boson: $d_g = 1-1/n$.

As an illustration of these remarks we now consider the most-studied two-dimensional critical point with non-Abelian [SU(2)] symmetry. This is the six-vertex model with isotropic vertex weights $V_{kl}^{ij} = \delta_k^i\delta_l^j + \lambda\delta_l^i\delta_k^j$ ($i=1,2$ denotes the direction of the arrow). This is equivalent to the $q=4$ Potts model.[11] Its transfer matrix[12] gives the isotropic spin-$\frac{1}{2}$ chain, $H = \sum_n \mathbf{S}_n\cdot\mathbf{S}_{n+1}$. This, in turn, is equivalent to a lattice fermion system,[13] by the Jordan-Wigner transformation. In the continuum limit it becomes the Thirring or sine-Gordon model[6] at $\beta^2 = 8\pi$. This is equivalent to the Coulomb-gas or xy model at the critical temperature.[14,15] We shall use the quantum spin-chain formulation as our starting point. The original derivation of the critical exponents[13] relies on the sequence of transformations mentioned above. The SU(2) symmetry becomes concealed at the first step (Jordan-Wigner transformation to a *single* species of fermions). The value of β in the sine-Gordon formulation is determined by uncontrollable renormalizations and, in fact, was found indirectly by comparison with the Bethe-*Ansatz* solution.[12] Needless to say, the *additional* SU(2) symmetry which appears in the critical theory was completely hidden.

We begin[16] by an exact transformation of the spin chain to a lattice fermion system which preserves the full symmetry. Introducing a spin doublet ψ_n^i we write $\mathbf{S}_n = \frac{1}{2}\psi_n^{\dagger i}\boldsymbol{\sigma}_i^j\psi_{nj}$ (sums over $i,j=1,2$ are implied). The standard fermion anticommutators $\{\psi_n^{\dagger i}, \psi_{mj}\} = \delta_{nm}\delta_j^i$, $\{\psi,\psi\} = \{\psi^\dagger,\psi^\dagger\} = 0$ reproduce the correct spin commutators. However, the Hilbert space of the fermion system is too large; we must project out states with one particle per site: $\psi_n^{\dagger i}\psi_{ni} = 1$. We now pass to the continuum limit. The standard way of doing this is[13,16] to retain only the Fourier modes of ψ_i with momentum $k \simeq \pm\pi/2a$ (where a is the lattice spacing). This is motivated by the assumption that only states close to the free-fermion ground state (in which all states with $|k| < \pi/2a$ are occupied) are important, or equivalently that only Fourier components of $\mathbf{S}_n$ with $k \simeq 0$, π/a, are important in the low-energy limit. Thus we introduce two new fermionic variables, $\psi_L(2n+\frac{1}{2})$, $\psi_R(2n+\frac{1}{2})$ by the exact transformation

$$\psi_j(n) = \sqrt{a}\,[i^n\psi_{jL}(n\pm\tfrac{1}{2}) + (-i)^n\psi_{Rj}(n\pm\tfrac{1}{2})]$$

(plus and minus for even and odd, respectively) which preserves the canonical anticommutation relation. We will use the notation of Eq. (6) for fermion bilinears and also $G = \psi_L^{\dagger i}\psi_{Ri} + \psi_R^{\dagger i}\psi_{Li}$, $G^a = \psi_L^\dagger\sigma^a\psi_R + \psi_R^\dagger\sigma^a\psi_L$. Then the constraint on even and odd sites becomes $J + \bar{J} = G = 0$. The spin variables become $S^a/a = (J^a + \bar{J}^a) + (-1)^n G^a$. Note that the $J,\bar{J}$ Schwinger terms given by Eq. (3) cancel *at equal times* in the commutator of the spin variables, as required. The constraint also implies $\mathbf{S}^2 = \frac{3}{4}$ or $(J^a + \bar{J}^a)^2 + (G^a)^2 = \frac{3}{4}a^2$. The Hamiltonian becomes

$$H = (a^2/2)\sum_n[(J^a+\bar{J}^a) \pm G^a](n\pm\tfrac{1}{2})[(J^a+\bar{J}^a) \mp G^a](n+1\mp\tfrac{1}{2}),$$

(upper and lower signs for n even and odd, respectively). Finally, we take the continuum limit by treating J^a, $\bar{J}^a$, and G^a as slowly varying. If we drop derivative terms and use the constraint this gives

$$H = (a/2)\int dx\,[J^aJ^a + \bar{J}^a\bar{J}^a + 2J\bar{J}^a]. \tag{7}$$

The momentum operator is simply that of the free-fermion theory. We now apply the non-Abelian bosonization rules of Eqs. (4)–(6). The constraints become $\partial_0\phi = \mathrm{Tr}g\cos[(2\pi)^{1/2}\phi] = 0$. These are satisfied by $\phi(x,t) = (\pi/8)^{1/2}$, with g arbitrary. The U(1) boson ϕ is "frozen" but the SU(2) field g is completely unconstrained. This is very natural, because the formulation of the spin chain in terms of spin-$\frac{1}{2}$ fermions introduced an additional, spurious U(1) "charge" symmetry. The constraint breaks this symmetry and, in the continuum limit, simply eliminates the corresponding charge boson. H and P become precisely that of the SU(2), $k=1$ Wess-Zumino model (after we set $J = \bar{J} = 0$) except for the interaction term $2J^a\bar{J}^a$ in H. But this term corresponds precisely to a Lorentz-invariant interaction Lagrangean $(\lambda/4)(\bar{\psi}\gamma_\mu\sigma^a\psi)^2$ or $(\lambda/4)\,\mathrm{Tr}g + \partial_- g\sigma^a\mathrm{Tr}\,\partial_+ gg + \sigma^a$ in bosonized form. The one-loop β function can be easily calculated (in fermionic language). We find $d\lambda/d\ln L = -(\pi/2)\lambda^2$, where L is the length scale. [Freezing of the U(1) boson does not change this β function because the interaction involves only the spin currents.] The spin chain has $\lambda \simeq 1$. Note that λ has the "wrong" (nonasymptotically free) sign. Thus λ renormalizes to zero at large length scales $\lambda(L) \sim 1/\ln L$, if we assume that no additional fixed points intervene between $\lambda = 0$ and $\lambda = 1$. *The critical behavior of the spin-$\frac{1}{2}$ chain is given by the theory of two free fermions with the free charge boson removed, and this is precisely the $k=1$, SU(2) Wess-Zumino model.*[17] Note that the theory of Eq. (7) has only a single SU(2) symmetry but the interaction term $(J,\bar{J})$ which breaks the chiral SU(2) is irrelevant and so the full SU(2) ⊗ SU(2) symmetry of the Wess-Zumino model emerges in the effective theory at the critical point.

We can now easily read off the critical behavior. After freezing of the charge boson, the spin variables become

$$S^a = a\,(J^a + \bar{J}^a) + (-1)^n a\,\mathrm{Tr}(g - g^\dagger)\sigma^a. \tag{8}$$

Thus the correlation function behaves as

$$\langle \mathbf{S}(n)\cdot\mathbf{S}(0)\rangle \sim \mathrm{const}/n^2 + (-1)^n\mathrm{const}/|n|.$$

The first term comes from the current-current correlation and scales as $1/n^2$ since the current has dimension 1; the second comes from the correlation function of g which has scaling dimension $\frac{1}{2}$ [see discussion below Eq. (6)]. Thus the correlation exponent is $\eta = 1$. The effect of adding various operators to the Hamiltonian can also be found. The only relevant operator that preserves the SU(2) symmetry is a staggered interaction, $H \to H + \gamma\sum_n(-1)^n\mathbf{S}_n\cdot\mathbf{S}_{n+1}$. This produces a cross term:

$$\gamma a^2\sum_{n,\pm}[J^a(n) + \bar{J}^a(n)]G^a(n\pm 1) \propto \gamma\sum_r \phi_L^{\dagger i}\phi_{Ri} - \phi_R^{\dagger i}\phi_{Li} + \text{(less relevant operators)}.$$

(The second form can be obtained by extending the fermion normal ordering to the four-fermion operator and thus obtaining quadratic terms.) Finally, when we use the bosonization formula and drop the charge boson this becomes simply

$$H \rightarrow H + \gamma\,\text{const} \int dx\, \text{Tr}g + (\text{less relevant}).$$

Another way of deriving this is to observe that the staggered interaction breaks the symmetry under translation by one lattice site. We see from Eq. (8) that this corresponds to the $g \rightarrow -g$ symmetry of the Wess-Zumino model. Trg is the most relevant operator that breaks this symmetry. Note also that it breaks chiral SU(2) leaving only the diagonal SU(2). Since Trg has scaling dimension $\frac{1}{2}$ we conclude that the theory develops a mass gas with $m \sim \gamma^{2/3}$. A staggered external field, $H \rightarrow H + h\sum(-)^n S_n^z$, similarly produces $\text{Tr}g\sigma^z$ and thus again $m \propto h^{2/3}$. A nonstaggered field produces $(J^z + \bar{J}^z)$. This is a redundant operator since it can be absorbed into a shift of the currents by a constant. A nonisotropic coupling, $H \rightarrow H + (\Delta - 1)\sum_n S_n^z S_{n+1}^z$, leads to $H \rightarrow H + (\Delta - 1)\int dx (J^z + \bar{J}^z)^2$. This is a marginal operator (dimension 2), and thus, with the appropriate sign, produces an exponentially small mass gap, the Kosterlitz-Thouless transition.[14] We note[8] that the possible Virasoro central charges for SU(2) are $c = 3k/(k+2) = 1, \frac{3}{2}, \frac{9}{5}, \ldots 3$. In the special case $k = 1$ we find $c = 1$, the same values as for the U(1) case. Thus if the SU(2) symmetry is broken to U(1) the critical theory may evolve continuously. All of these conclusions are in complete agreement with those found previously.[6,13,17,18] The present approach exposes the source of marginal operators (responsible for the Kosterlitz-Thouless transition) as the existence of conserved currents and makes clear the full symmetry of the critical theory. Furthermore, these methods can be readily extended[19] to chains of higher spin and higher symmetry and hence to the quantum Hall-effect localization transition.[20]

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