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Evidence for Ising-Type Critical Phenomena in Two-Dimensional Percolation

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A recent theory of two-dimensional percolation predicts critical singularities identical to those appropriate for the associated dilute Ising model. This implies the absence of a separate universality class for two-dimensional percolation processes. Novel numerical and series-expansion studies are presented in support of this unexpected result.

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The concept of percolation is fundamental to our understanding of the properties of disordered systems. Anomalies in the structural and transport properties of random media are often associated with corresponding singular behavior in the geometrical properties of an underlying percolation problem.[1]

Over the last decade, a consistent picture of the critical behavior near the percolation threshold has emerged[2] as the result of approximate theories which rely on the (assumed) standard forms of the singularities. In this picture, the percolation process is in a universality class of its own and in two dimensions (2D) the exponents are thought to be known exactly.[3]

In a recent Letter,[4] I have proposed a novel theoretical tool, based on Grassmann path integrals (GPIs), for investigation of the behavior at phase transitions in 2D Ising-spin models. As is well known,[1–3] a percolation problem corresponds to the zero-temperature limit of an Ising model on the diluted lattice which sustains the percolation process (or, alternatively, to the $q=1$ limit of a q-state Potts model on the undiluted lattice). In a GPI theory, the 2D bond-dilute Ising model near criticality has been shown[4] to be described by an effective action which in terms of the $n \to 0$ Grassmann fields ψ^a reads

$$S_{\text{eff}} = \int d^2x \left[\frac{1}{2} i \sum_{a=1}^{\mu} \bar{\psi}^a (m + \boldsymbol{\partial}) \psi^a - g \left(\sum_a \bar{\psi}^a \psi^a \right)^2 \right], \tag{1}$$

where $m \propto pt - (pt)_c$, $g \propto (1-p)$, p is the concentration of bonds, and $t = \tanh(\beta J)$ is the usual Ising thermal variable (the subscript c denotes the critical point). I have shown that an exact GPI renormalization-group (RG) treatment for the action of Eq. (1) predicts for the singular part of the Ising free energy the form

$$-\beta f_s \sim |m|^2 \ln|\ln|m||. \tag{2}$$

Accordingly, if $p\,(>p_c)$ is kept fixed, then the singular thermal behavior $f_s(T) \sim |T - T_c(p)|^2 \ln|\ln|T - T_c(p)||$ is predicted. If, on the other hand, T is fixed at $T=0$, then Eq. (2) yields the following singular form for the percolation's mean number of clusters per site,[2] $K(p)$:

$$K_s(p) \sim |p - p_c|^2 \ln|\ln|p - p_c||, \tag{3}$$

both for $p \to p_c^+$ and $p \to p_c^-$. This result is to be contrasted with the currently accepted form of the singularity,[1–3]

$$K_s(p) \sim |p - p_c|^{2-\alpha}, \tag{4}$$

with $\alpha = -\frac{2}{3}$. This value of α follows from the use of the hyperscaling relation, $2 - \alpha = d\nu$, and of the conjectured exact value of the correlation length exponent $\nu = \frac{4}{3}$. The GPI prediction, Eq. (3), implies on the other hand that $\alpha = 0$, just as for the dilute Ising model. Also, the fact that no new symmetry or singular temperature dependence arises in the action of Eq.

(1) as $T \to 0$ strongly suggests that the critical behavior of the remaining properties of 2D percolation will likewise be identical to the behavior of the corresponding functions of the dilute Ising model. The latter is yet unknown in detail from use of, say, the GPI approach, but my expectation is that exponents will be those of the 2D pure Ising model with superimposed marginal corrections. In other words, Eq. (1) implies that the percolation limit of the 2D dilute Ising model is not a multicritical point and that 2D percolation does not represent a separate universality class.

In this Letter, I will give evidence that these unexpected predictions may indeed be correct. In particular, I will present novel numerical and series-expansion work that supports the GPI prediction, Eq. (3), and rules out the conventional form, Eq. (4), for $K(p)$.

I begin with the series-expansion studies. A low-density nineteen-term expansion for $K(p)$ is available[5] for the square-lattice bond percolation problem, for which $p_c = \frac{1}{2}$. This series has been analyzed by Domb and Pearce (DP)[6] who assumed the form of Eq. (4) for $K(p)$. Their ratio-test analysis, reported in Fig. 1, strongly indicates convergence toward the value $\alpha = -\frac{2}{3}$. However, a similar convergence arises when the following model series is analyzed in the same fashion and with the assumption of the form of Eq. (4):

$$K(p) = (1-p/p_c)^2 \ln[1 + C\ln(1-p/p_c)]F(p) = \textstyle\sum_n K_n p^n, \tag{5}$$

with C an adjustable constant and $F(p) = (1+p/u_0 + p^2/u_0)^y$ a factor mimicking the superimposed complex singularity[6] present at $u_0 = -p_0(1+p_0) = 0.259$ with $y = 0.3$ in the original series for $K(p)$. I have found that, although technically $\alpha = 0$, the ratio-test analysis for Eq. (5) initially appears to converge toward a value of $\alpha \sim -0.4$ which is virtually C independent and is close to $-\frac{2}{3}$. Convergence toward this fictitious value of α is as good as in the case of the original analysis of DP, although not quite the same since Eq. (5) models the leading singular behavior of $K(p)$ only. As the presence of a complicated logarithmic correction leads to misleading conclusions on the value of an assumed power-law exponent, a different type of series analysis must be sought. The ideal candidate is the method of Adler and Privman,[7] which is specifically designed for the analysis of logarithmic corrections. One assumes that $K(p)$ presents the singular form

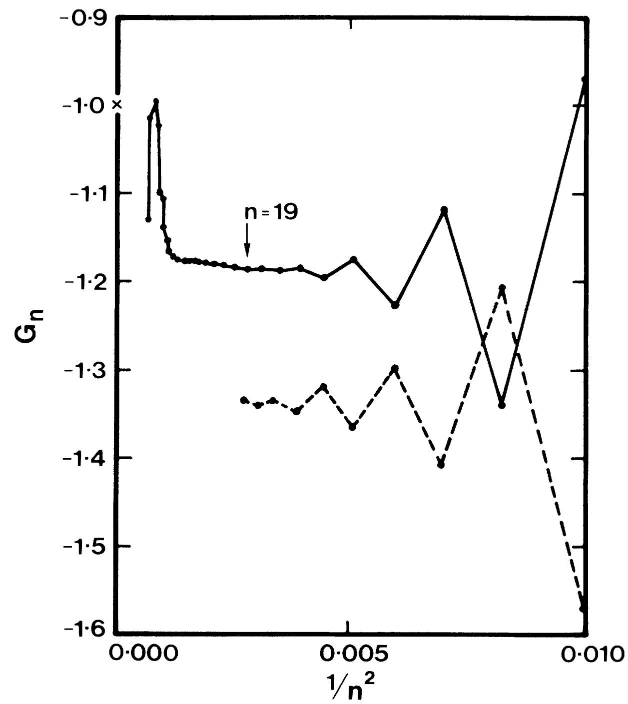

FIG. 1. First-order Neville-table ratio test for $K(p)$; for $n \gg 1$, $G_n = -1 + \alpha/2 + O(1/n^2)$. Dashed line: original analysis of Domb and Pearce. Solid line: present analysis for model series Eq. (5) with $C = -1.25$; cross denotes true asymptotic value ($\alpha = 0$).

$$K_s(p) = c(p)(p_c - p)^h [\ln(p_c - p)]^{zh},$$

with $h = 2 - \alpha$ and $c(p)$ incorporating all corrections to the leading behavior. An analytic background term $b(p) = K(p) - K_s(p)$ is subtracted from $K(p)$ in what follows; in the absence of further information, $b(p)$ is chosen in the form predicted by DP.[6] Then, Padé approximants to the series

$$g(p) = h^{-1}(p - p_c)\ln(p_c - p)[K_s'(p)/K_s(p) + h(p_c - p)],$$

will give the exponent $z = z(h) = g(p_c)$ as a function of the input value of h. For the percolation problem at hand, my analysis is given in Fig. 2 and one can see that the expected result, $z = 0$ for $h = 2 + \frac{2}{3}$, is not attained, whereas the GPI result, $z = 0$ for $h = 2$, is supported. Similarly, I have repeated the analysis with an assumed singular form

$$K_s(p) = c(p)(p_c - p)^h \{\ln[-\ln(p_c - p)]\}^{\hat{z}h}, \tag{6}$$

and made use of

$$\hat{g}(p) = h^{-1}(p - p_c)\ln(p_c - p)\ln[-\ln(p_c - p)][K_s'(p)/K_s(p) + h/(p_c - p)], \tag{7}$$

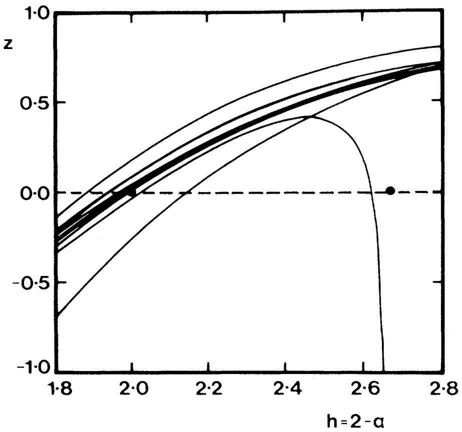

FIG. 2. Central Padé approximants in the search for logarithmic corrections in $K(p)$. Dots denote competing theoretical values (see text).

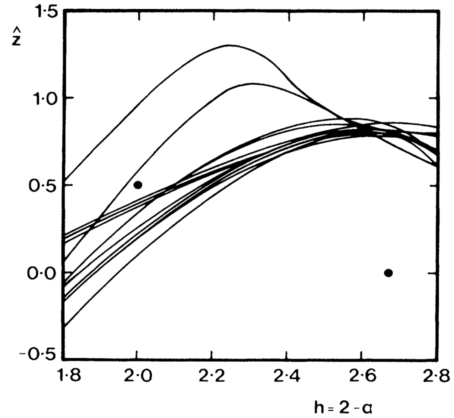

FIG. 3. Central Padé approximants in the search for log-log corrections in $K(p)$. Dots denote competing theoretical values (see text).

which gives $\hat{z}=\hat{z}(h)=\hat{g}(p_c)$. The analysis for Eqs. (6) and (7) is presented in Fig. 3; again, one can see that the expected result, $\hat{z}=0$ for $h=2+\frac{2}{3}$, is disproved while the GPI prediction, $\hat{z}=\frac{1}{2}$ for $h=2$, is confirmed. Very similar results have been obtained for the site percolation problem on the triangular lattice.

Next, I discuss an alternative numerical study of the singularity in $K(p)$. Because of the weak nature of this singularity, the quantity one looks at numerically is $K'''(p)=d^3K(p)/dp^3$. For a lattice of size L, one expects a divergence in $K'''(L,p_c)$ as $L \to \infty$. Finite-size scaling analysis[8] yields the prediction

$$K'''(L,p_c) \sim AL^{(1+\alpha)/\nu}+B, \tag{8}$$

for $L >> 1$, with A and B constants. Therefore, the conventional exponents of 2D percolation would predict the asymptotic behavior $K'''(L,p_c) \sim L^{1/4}$. If on the other hand one takes the prediction of the GPI theory, one expects

$$K'''(L,p_c) \sim AL^{1/\nu}l(L)+B, \tag{9}$$

where $l(L)$ is yet unknown and contains logarithmic corrections. In order to evaluate $K'''(L,p_c)$ numerically, I have made use of the fluctuation formula

$$\begin{aligned} K'''(L,p_c) = {} & 32(\langle n_c N_l\rangle - \langle n_c\rangle\langle N_l\rangle) + 64(\langle n_c N_l^3\rangle - 3\langle n_c N_l^2\rangle\langle N_l\rangle + 6\langle n_c N_l\rangle\langle N_l\rangle^2 \\ & - 6\langle n_c\rangle\langle N_l\rangle^3 + 6\langle n_c\rangle\langle N_l^2\rangle\langle N_l\rangle - 3\langle n_c N_l\rangle\langle N_l^2\rangle - \langle n_c\rangle\langle N_l^3\rangle), \end{aligned} \tag{10}$$

where angular brackets denote configurational averages, and n_c and N_l are the number of clusters per site and the number of links for each configuration, respectively. The noise generated by Eq. (10) requires averaging with respect to a considerable number of configurations, thus limiting the value of L. In Fig. 4, preliminary results are shown for the square-lattice bond percolation problem; the maximum size available is $L=30$, and this lattice alone required 2×10^7 configurations altogether, which correspond to 300 hours CPU time on a Cray-1 computer. Four sets of data are given in Fig. 4. Triangles refer to a $K(p)$ which includes linked clusters only; circles refer to $K(p)$ for all clusters, including unlinked sites. Open circles and open triangles refer to the true $K'''(L,p_c)$, for which the statistical noise becomes rapidly uncontrollable. Filled circles and filled triangles refer to a modified counting algorithm which was found to be considerably less noisy and in which noncritical clusters are partly neglected in the counting. Presumably, this alogrithm evaluates $\tilde{K}(p)=K(p)-H(p)$, where $\tilde{K}(p)$ and $K(p)$ have the same singular behavior. If one assumes that the asymptotic behavior of $L >> 1$ is reached in these enumerations on small lattices [the initial curvature of data in Fig. 4 being due mostly to the constant B in Eqs. (8) and (9)], then on all sets of data one consistently reads a slope $(1+\alpha)/\nu=1$. This is far from the slope of $\frac{1}{4}$ predicted by the conventional theory and Eq. (8). The numerical data suggest, within the limitations of the method, agreement with the GPI prediction, Eq. (9), and consistency with an effective exponent $\nu=1$ for the correlation length. This result would fit the naive expectations of the GPI theory, as well as the hypothesis of hyperscaling.

In conclusion, I have presented novel numerical and

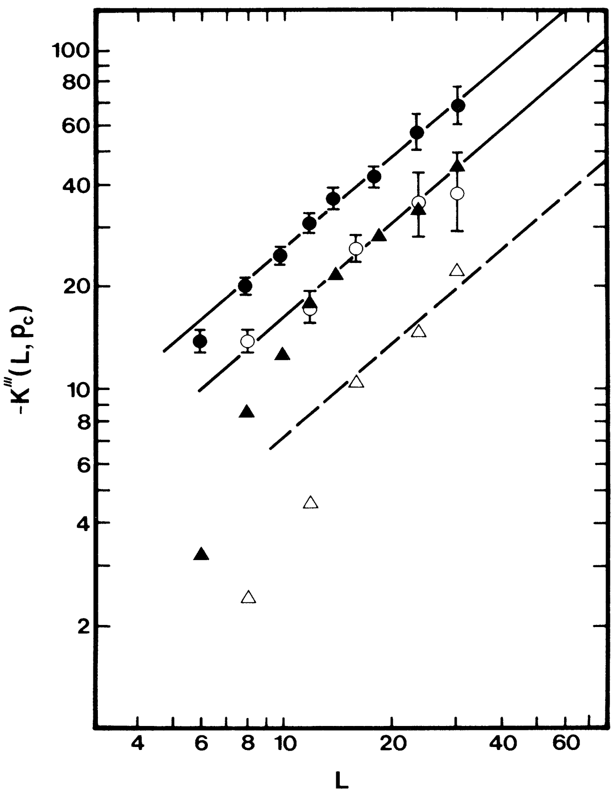

FIG. 4. Divergence of $K'''(L,p_c)$ with lattice size L. See text for explanation of symbols. Straight lines have slope = 1 and are guides to the eye for the presumed asymptotic behavior $L >> 1$.

series-expansion analysis for the singularity of 2D percolation's mean number of clusters, $K(p)$. There is some evidence that the GPI prediction, Eq. (3), is correct. It is not clear at present whether this represents merely a breakdown of hyperscaling or whether the entire scaling theory of 2D percolation is at fault. Certainly, I expect more marginal corrections to appear in the remaining properties of 2D percolation and there is a strong possibility that these corrections have deceived any existing approximate theory. Incidentally, trouble with the current view of 2D percolation has been anticipated by different authors in the past. Parisi and Fucito[9] have found evidence for the breakdown of the RG $6-\epsilon$ expansion for percolation specifically in 2D. Andelman and Berker[10] have found real-space RG evidence for a marginal operator (thus, logarithmic corrections) in the 2D $q=1$ Potts model. Although there is no conclusive evidence as yet that the accepted theory of 2D percolation is entirely inadequate, my hope is that this Letter will stimulate the interest of other workers on the important issues hereby presented.

I have benefitted from discussions with a number of colleagues; I am indebted to H. A. Duncan and V. Privman for advice which has proven to be invaluable for the success of this work. I am grateful to M. F. Sykes for providing me with his unpublished coefficients for the series expansion for $K(p)$. My thanks are due to R. R. Price and Westinghouse Electric Corporation for providing time on their Cray-1 computer which has made the completion of this project possible. Much of this work has been carried out while I was in Pittsburgh and I thank the National Science Foundation for support through Grant No. DMR-8302326.

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