Range of Validity of the Einstein Relation

In a recent Letter[1] a relation[2] between the fractal dimension d_f, the fracton dimension d_s, and the resistivity exponent x was considered:

$$d_f = d_s x/(2-d_s). \tag{1}$$

The author concludes from this relation that (i) when $d_s \to 2$, $d_f \to \infty$ and the structure is collapsed in all dimensions, and (ii) for $d_s > 2$ the Einstein relation[2]

$$d_w = d_f + x \tag{2}$$

ceases to apply.

However, we argue here that d_f can be finite when $d_s \to 2$ and the Einstein relation [Eq. (2)] is valid also for $d_s \geqslant 2$. This will be the case if the exponent x in Eqs. (1) and (2) is interpreted as the exponent characterizing the resistance between the two bars[3-5] of size r^{d-1} separated by a distance r and not as interpreted by Ref. 1 as the resistance between two sites. Both exponents are the same for finitely ramified fractals.[6] However, for infinitely ramified aggregates characterized by $d_s \geqslant 2$ the two exponents are not the same in general.[6] This interpretation of x allows $x=0$ for $d_s=2$ and $d_w=d_f=$finite. Also for $d_s>2$ it allows $x<0$, so that Eq. (2) is still valid.

We present three examples for which $d_s \geqslant 2$ where Eqs. (1) and (2) are valid and d_f is finite. The simplest example is provided by compact clusters of fractal dimension $d_f=d$. For this case clearly $d_w=2$, $x=2-d$, and from $d_s=2d_f/d_w$ it follows that $d_s=d$ and both Eqs. (1) and (2) are valid. Another example is the family of exact fractals studied in Ref. 3. This family includes the case $d_s=2$, for which d_f is found to be finite. Finally, in the following we present a family of random clusters with $d_s \geqslant 2$ for which Eqs. (1) and (2) are valid and d_f is finite when we use our interpretation of x.

The model is a family of random clusters, without loops (trees) and without dead ends, embedded on a Cayley tree with coordination number $n=3$. In this model we generate trees with adjustable $d_f \geqslant 2$. Let $P(l)=\alpha/l$ be the probability that a site in generation $l-1$ will grow to two sites in generation l, and $1-P(l)$ be the probability that it will grow to only one site. The expected number of sites that grow from one site in the $(l-1)$st generation is $2\times P(l)+1\times[1-P(l)]=1+P(l)$. Thus the total number of sites $B(l)$ in the lth generation is

$$B(l)=\prod_{l'=1}^{l}[1+P(l')]=l^{\alpha}, \quad l>>1, \tag{3}$$

from which the total mass is $M(l)\sim\sum_{l'=1}^{l}B(l') \sim l^{\alpha+1}\equiv l^{d_l}$. Since trees grown on a Cayley tree have the property that $r\sim\sqrt{l}$, it follows that $d_f=2d_l=2(\alpha+1)\geqslant 2$ $(\alpha\geqslant 0)$. The diffusion on these trees was calculated exactly[6] and found to be $\langle l^2\rangle\sim t$. Thus one finds $\langle r^4\rangle\sim t$ or $d_w=4$ and $d_s=2d_f/d_w=d_f/2=d_l$. This result, $d_w=4$, can be obtained also from the Einstein relation, Eq. (2). Let $\rho(l)$ be the resistance between generation 1 and all sites in generation l, and let $\rho_1(l)$ be the resistance between one site in generation l and a single site in $l=1$. Then $\rho(l)=\rho_1(l)/l^{d_l-1}$, since l^{d_l-1} represents the effective number of parallel paths to the lth shell. But clearly $\rho_1(l)\sim l$ (no loops!), so that $\rho(l) \sim R^2/R^{2(d_l-1)}\sim R^{4-d_f}\sim R^x$. Substituting $x=4-d_f$ in Eq. (2) we obtain $d_w=4$ as found exactly, independently.[6] Notice that if these fractals are embedded in a dimension d, satisfying $d<d_c$ (d_c is the upper critical dimension for this model) then

$$d_w=2d_f/d_l, \quad d_s=d_l. \tag{4}$$

These results are generalizations of the results of diffusion on linear chains[2] which are obtained when one substitutes $d_l=1$ in Eqs. (4). It should also be noted that Eqs. (4) and the results $d_w=d_f(1+1/d_l)$, $d_s=2d_l/(d_l+1)$ found recently for finitely ramified clusters without loops are special cases of a general relation given by Havlin *et al.*[7]

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