Period-Doubling Systems as Small-Signal Amplifiers

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Near the onset of a period-doubling bifurcation, any dynamical system can be used to amplify perturbations near half the fundamental frequency: The closer the bifurcation point, the greater the amplification. An analytic expression for the frequency response curve is derived explicitly for the driven Duffing oscillator. Results of analog simulations are presented to check the main features of the theory. We propose that the superconducting Josephson parametric amplifier is an example of this amplification process.

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A great many systems, representing a wide variety of physical phenomena, are known to undergo period-doubling instabilities. The goal of nonlinear dynamics is to determine what such systems have in common, irrespective of differences in the underlying physics. The most familiar results about period-doubling bifurcations concern the so-called period-doubling cascade, in which an *infinite* sequence of instabilities occur, culminating in chaotic dynamics. Most of the theoretical work has emphasized the dynamical behavior close to the onset of chaos, or within the chaotic regime. Researchers have found that several quantities obey scaling laws[1-8] which are reminiscent (formally, at least) of scaling behavior observed in condensed-matter critical phenomena. On the experimental side, period-doubling sequences have proven to be fairly common, having been observed in electrical,[9-14] optical,[15] hydrodynamic,[16] chemical,[16] and biological systems.[17,18]

This Letter concerns the behavior of dynamical systems near the onset of a *single* period-doubling instability, far from a chaotic regime. This topic has received less attention then the period-doubling cascade, despite the fact that the mathematics of bifurcation theory tells us a great deal about the dynamics in such a situation.[19] Recently, work on the effect of random noise as a single instability is approached has shown that new broadband lines are induced in the power spectrum.[20,21] In effect, the input noise is greatly amplified, before the bifurcation, near frequencies where new sharp spectral lines appear *after* the bifurcation. Measurements on driven *p-n* junctions[20] are in excellent agreement with the theoretical predictions.[21]

The amplification of broadband noise at certain frequencies suggests that small *coherent* perturbations might also be amplified by systems near the onset of a period doubling. The purpose of this Letter is to show that this is indeed the case.

Our basic result is as follows. Consider any dynamical system oscillating with period T, and suppose a parameter is adjusted so that the system is just before the onset of a period-doubling bifurcation. If one now couples in a small, monochromatic perturbation at frequency $\omega = \pi / T$, the output of the system will have a large component at ω. The magnitude of this amplification will grow substantially as the bifurcation point is approached.

This result is independent of the specific dynamical system studied; however, to illustrate the ideas involved we focus on a specific example—the driven Duffing oscillator,

$$\ddot{x} + \gamma \dot{x} + \alpha x + \beta x^3 = A + B \cos t, \tag{1}$$

where $\alpha, \beta, \gamma > 0$. The main advantages of focusing on this system are (1) that the derivation of the basic results can be kept reasonably brief, and (2) that we can test these results easily on an analog computer (see Fig. 1). As will be mentioned later, the deriva-

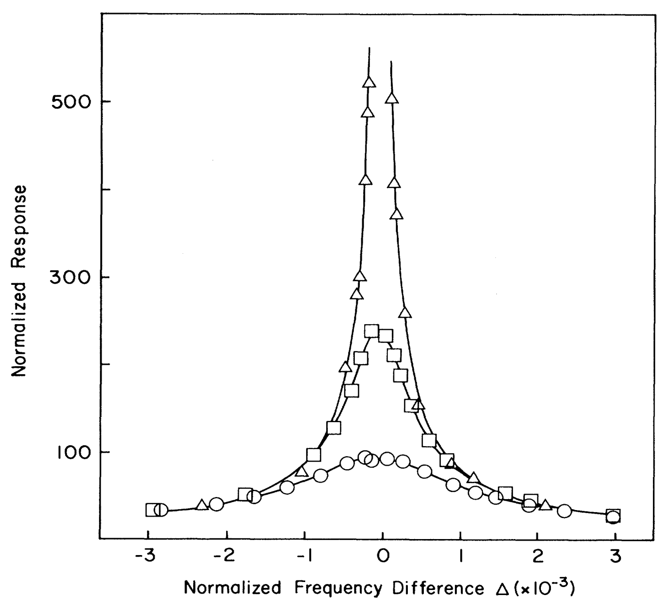

FIG. 1. Amplitude response $V(\omega)$ vs signal detuning frequency Δ, for analog simulation of Eq. (2). Data are shown for three different parameter values, just before the onset of a period-doubling bifurcation. The circles, squares, and triangles correspond to successively smaller bifurcation parameters ϵ.

tion can be generalized in several ways. Full details of a general derivation will be presented in a longer communication.[22]

Equation (1) describes the motion of a damped, driven particle subject to a single-well, anharmonic potential. For small driving amplitude B, the particle executes 2π-periodic motion. As B is increased, the system undergoes a succession of period-doubling instabilities, until the motion becomes chaotic.[12] We want to focus on the behavior of $x(t)$ when the system is tuned just before the onset of the first period doubling, and a second, low-amplitude driving force is added at a frequency near the half-fundamental:

$$\ddot{x}+\gamma\dot{x}+\alpha x+\beta x^3=A+B\cos t+\lambda\cos\omega t. \tag{2}$$

Let $x_0(t)$ be a stable 2π-periodic solution to the unperturbed system (1). Then for small enough λ, the deviation $\eta=x-x_0$ will be governed by Eq. (2) linearized about x_0:

$$\ddot{\eta}+\gamma\dot{\eta}+[\alpha+3\beta x_0^2]\eta=\lambda\cos\omega t. \tag{3}$$

We solve this using a Green's-function approach, so that

$$\eta(t)=\int_0^t G(t,t')\lambda\cos\omega t'\,dt', \tag{4}$$

where G satisfies Eq. (3) with the right-hand side replaced by the impulse $\delta(t-t')$. Since this equation is linear with periodic coefficients, an explicit construction of G can be performed by use of the results of Floquet theory.[23] In particular, G is the sum of special solutions χ_k of the homogeneous problem, such that

$$\chi_k(t,t')=e^{\rho_k(t-t')}P_k(t)Q_k(t'),\quad k=1,2,$$

where P_k, Q_k are 2π-periodic functions.

Physically, the χ_k represent the response of the (linearized) system to an impulse forcing at $t=t'$. Stability of x_0 requires that the response η be transient, so that the Floquet exponents satisfy $\mathrm{Re}\rho_k<0$. A period-doubling bifurcation occurs when one exponent crosses the imaginary axis at $\rho=i/2$. Precisely at the bifurcation point, one of the exponents (ρ_1, say) is equal to $i/2$, and one sees from the last equation that χ_1 is indeed 4π-periodic. *Near* the bifurcation, $\rho_1=-\epsilon+i/2$ with ϵ a very small positive number, and $\mathrm{Re}\rho_1/\mathrm{Re}\rho_2<<1$. Consequently, G is dominated by χ_1. If we neglect the contribution of the relatively short-lived transient χ_2, Eq. (4) becomes

$$\eta(t)=\int_0^t e^{-\epsilon(t-t')}p(t)q(t')\lambda\cos\omega t'\,dt', \tag{5}$$

where

$$p(t)=e^{it/2}P_1(t),\quad q(t')=e^{-it'/2}Q_1(t'),$$

so that $p(t+2\pi)=-p(t)$ and $q(t'+2\pi)=-q(t')$. This last property implies that the Fourier expansions for p and q contain only odd harmonics:

$$p(t)=\sum_{n\ \mathrm{odd}} a_n e^{int/2},\quad a_{-n}=a_n^*;$$

$$q(t')=\sum_{m\ \mathrm{odd}} b_m e^{imt/2},\quad b_{-m}=b_m^*.$$

The integral appearing in Eq. (5) is readily evaluated, since the integrand is the product of exponential and periodic functions. With use of the expansions for p and q, Eq. (5) becomes

$$\eta(t)=\int_0^t e^{-\epsilon(t-t')}\sum_{n,m} a_n b_m e^{i(nt+mt')/2}\lambda\cos\omega t'\,dt'.$$

For $\omega=\frac{1}{2}+\Delta$, with Δ small, only the $m=\pm 1$ terms contribute significantly, with the result

$$\eta(t)=\sum_{n\ \mathrm{odd}}\tfrac{1}{2}\lambda a_n e^{int/2}\left[\frac{b_1 e^{-i\Delta t}}{(\epsilon-i\Delta)}+\mathrm{c.c.}\right],$$

where we have also taken $\epsilon t>>1$. This is a discrete Fourier sum, and so the power spectrum is a sequence of δ-functions:

$$S(\Omega)=\sum_{n\ \mathrm{odd}}\frac{\lambda^2|a_n b_1|^2}{4(\epsilon^2+\Delta^2)}\left\{\delta\left[\Omega-\frac{n}{2}-\Delta\right]+\delta\left[\Omega-\frac{n}{2}+\Delta\right]\right\}. \tag{6}$$

Of course, the full power spectrum also includes the sequence of δ functions at integer frequencies due to the basic oscillation x_0.

As the period doubling is approached, $\epsilon\rightarrow 0$, and a very large response at the signal frequency occurs for small detuning Δ. Note that strong responses also occur at frequencies $n/2+\Delta$, for n odd. Had the signal been near a different half-integer frequency $\omega=m/2+\Delta$ (m odd), Eq. (6) would hold with b_m replacing b_1.

After the onset of the period-doubling bifurcation, $\mathrm{Re}\rho_1>0$ and the 2π-periodic solution x_0 is no longer stable. Equation (2) must now be analyzed by linearization about the new, stable 4π-periodic solution $\tilde{x}$. Just after the bifurcation, one finds that one of the new Floquet exponents is $\tilde{\rho}_1=-\tilde{\epsilon}$, where $\tilde{\epsilon}$ is a small positive number—that is, the system is near the bifurcation point of a different type of instability (i.e., a pitchfork bifurcation). A calculation[22] of the power spectrum for this situation reveals that $S(\Omega)$ is again a Lorentzian, varying as $(\tilde{\epsilon}^2+\Delta^2)^{-1}$.

We emphasize that the derivation of Eq. (6) is based on the linearized Eq. (3), which is valid for $|\eta|<<|x_0|$. For small enough ϵ and Δ, however, the response η can be relatively large, and a proper analysis must include the nonlinear terms ignored in Eq. (3).

To test Eq. (6), we have implemented the driven Duffing oscillator on an analog computer. Having set the parameters so that the system (1) is close to the onset of the first period doubling, we added a small signal with $\omega=\frac{3}{2}+\Delta$. In Fig. 1 we plot the linear spectrum $V(\omega)=\sqrt{S}(\omega)$ versus frequency difference Δ, for three different values of bifurcation parameter ϵ. The data have been normalized to the input-signal strength, so that the ordinate is the factor by which the input-signal amplitude is amplified. (The *power* gain is given by the square of the ordinate.) Qualitatively, we see that the response curves grow and sharpen dramatically as ϵ decreases. Moreover, the data fit quite well to the square root of a Lorentzian (solid curves). If Eq. (6) is correct, the height-width product of these response curves should be independent of ϵ. Using the χ^2 fits shown, we obtain height-width products of 0.154, 0.150, and 0.146, for the circles, squares, and triangles, respectively.

The analysis based on Eq. (2) can be extended in three important ways.[22] First, the results extend to general dynamical systems $x=F(x)$ near a period doubling. Second, the unmodulated system may be a self-oscillating (i.e., autonomous) system. Third, the modulation $\lambda \cos\omega t$ need not enter additively, as in Eq. (2), but can enter instead as a *parametric* modulation. We also remark that an entirely analogous analysis holds near the onset of instabilities other than period doublings. For example, before a Hopf bifurcation, corresponding to an instability in which the (unperturbed) system begins to oscillate at a second independent frequency Ω, amplification of small signals at frequencies near Ω can occur.[22]

What we propose, then, is this: If one builds *any* dynamical system that oscillates with frequency f, and this system is tuned near a period-doubling instability, then coupling in of a small signal at a frequency near $f/2$ will result in a large response of the system at the signal frequency. In view of the generality of this result, and the large number of systems known to display period-doubling instabilities, this proposition could be tested quite readily by experiments.

What are the *practical* advantages of this method of amplification? This remains an open question; however, consider the superconducting devices called singly degenerate parametric amplifiers, based on the Josephson-junction technology.[24] Introduced a decade ago, these devices were proposed as attractive candidates for low-power microwave amplifiers.[25,26] Researchers have reported that the parameter conditions for a period-doubling instability are closely related to the conditions for good gain.[26–28] The theoretical understanding of this relationship is based on the specific analysis of the resistively shunted Josephson-junction model. Indeed, based on calculations like the one presented here, we can show[22] that the parameter values for maximum gain *coincide* with the onset of period doubling in the singly degenerate Josephson amplifiers. Viewed in a larger context, we claim that this is but one example of the general amplification mechanism discussed in this Letter.

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