Bistable Solitons

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It is demonstrated that a nonlinear Schrödinger equation with certain nonlinearities allows for an existence of multistable singular solitons (i.e., singular solitons with the same carried power but different propagation parameters). In nonlinear optics, these solitons may exist in the form of either short bistable pulses, or bistable self-trapping (both two- and three-dimensional).

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In this Letter we demonstrate that for a certain class of nonlinearities, the soliton solution of the (generalized) nonlinear Schrödinger equation becomes multistable. This implies that more than one amplitude profile and speed of propagation of a singular soliton may exist for the same amount of total power carried by the soliton. The existence of multistable solitons is related to the type of dependence of nonlinear susceptibility on the intensity of light. For example, the multistable soliton waves cannot be observed in a Kerr-type nonlinear medium; they may exist only either if the nonlinear component of the susceptibility as function of intensity is changing its sign or its derivative has a sufficiently sharp peak (e.g., it is a step-like function).

The soliton bistability may result in such effects as bistable (or multistable, in general) self-trapping of light in media with nonlinear refractive index, as well as bistable propagation of short soliton pulses in nonlinear optical fibers, since both of them may be described by the same nonlinear equation. Both of these effects may be viewed as an ultimate manifestation of multistable wave propagation since they are based on the simplest possible propagation configuration. They may also provide new opportunities in the field of optical bistability.[1] Indeed, for example, a bistable self-trapping of light provides a potential for an optical bistable device entirely free from any cavity or Fabry-Perot resonators,[1] single nonlinear interfaces[2] or nonlinear wave guides formed by the nonlinear interfaces,[3] retroflection self-action effects,[4] four-wave mixing,[5] etc. On the other hand, since the propagation of singular pulses in a homogeneous nonlinear medium[6,7] and in nonlinear fiber wave guides[8] is also governed by a nonlinear Schrödinger equation, these soliton pulses in the system with an appropriate nonlinearity may provide the first (to the best of our knowledge) known opportunity to attain a temporal (or dynamic) bistability as opposed to all known kinds of optical bistability which have been so far formulated in terms of steady-state regimes. The very notion of steady-state optical bistability comes into inevitable contradiction with the applications, most of which assume fast pulse regime of operations. When exploited in a dynamic regime, such effects still demonstrate hysteretic behavior which, however, can hardly be identified with the original "adiabatic," steady-state hysteresis. The dynamic hysteresis is more strongly affected by the relaxation processes than by steady-state bistable states, especially when the total switching cycle has the duration time of the same order as relaxation times. The truly dynamic (or temporal) bistability discussed in this paper is based on bistable pulse shapes (as well as on bistable duration of the pulses) and offers a way to resolve this contradiction.

Consider the generalized nonlinear Schrödinger equation for the complex amplitude of field E in the form

$$2i\partial E/\partial z+\partial^2 E/\partial x^2+Ef(|E|^2)=0, \tag{1}$$

where $f(|E|^2)$ is an arbitrary function of the intensity $|E|^2$ with $f(0)=0$. When $f(|E|)^2=\alpha|E|^2$ (α $=$const), Eq. (1) is the so-called cubic nonlinear Schrödinger equation[7,9–11] which corresponds to Kerr nonlinearity in optical propagation. In the case of two-dimensional self-trapping,[11] z is the normalized axis of soliton propagation and x is the normalized transverse axis (both of them are dimensionless and correspond to the real coordinates $\tilde{z}$ and $\tilde{x}$ multiplied by the wave number $k=\omega n/c$). In the case of one-dimensional pulse propagation along the z_1 axis in a slightly dispersive medium with a nonlinearity $f_1(|E|^2)$, the equation of propagation is[6–8,11]

$$2i\partial E/\partial z_1+(dv/d\omega)v^{-2}\partial^2 E/\partial\xi^2 + kEf_1(|E|^2)=0, \tag{1'}$$

where $\xi=t-z_1/v$; $v=d\omega/dk$ is the group velocity of linear propagation. Equation (1′) can readily be transformed into Eq. (1) by proper scaling, e.g., by assuming

$$z_1=(z/k^2)(dv/d\omega);\quad \xi=x/kv;$$
$$f_1=fk(dv/d\omega).$$

In both cases f is proportional to the nonlinear (i.e., intensity-dependent) component $\Delta\epsilon^{NL}$ of the dielectric constant ϵ of the medium. The nonlinear Schrödinger equation is obtained from the Maxwell equations in the conventional slowly varying envelope approximation (i.e., $\partial E/\partial z \ll \partial^2 E/\partial x^2$) which implies either small (quasioptical) diffraction [Eq. (1)] or relatively small dispersion [Eq. (1′)].

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The stationary solutions (in particular, singular solitons) of Eq. (1) have a nonvarying intensity profile, $\partial|E|^2/\partial z=0$, i.e., such solutions are written as $E(x,z)=u(x)\exp(i\delta z/2+i\phi)$, where $\phi=$ const and δ is the (unknown) real speed (or propagation constant) of the soliton. Thus the equation for the real amplitude $u(x)$ is

$$d^2u/dx^2+u[f(u^2)-\delta]=0, \tag{2}$$

whose soliton solution must satisfy the condition $u\to 0$ as $|x|\to\infty$ in order for the total power $P=\int_{-\infty}^{\infty}u^2\,dx$ to be limited. Under this condition the first integral of Eq. (2) is

$$(du/dx)^2=2\int_0^u u[\delta-f(u^2)]\,du, \tag{3}$$

integration of which yields

$$x=\int\left\{\int_0^{u^2}[\delta-f(u^2)]\,d(u^2)\right\}^{-1/2}du. \tag{3$'$}$$

This determines implicitly the soliton amplitude profile $u(x)$ for each particular δ and $f(u^2)$. The integral in Eq. (3′) can be analytically evaluated only for some particular class of functions $f(u^2)$, but this may not be done in the case of arbitrary $f(u^2)$. In order to evaluate a total power P, however, the explicit form of $u(x)$ does not need to be known. Indeed, by the use of Eq. (3) and the introduction of $I=u^2=|E|^2$, it is shown that

$$P(\delta)=\int_0^{I_m(\delta)}dI/[\delta-F(I)]^{1/2}, \tag{4}$$

where

$$F(I)=I^{-1}\int_0^I f(I)\,dI,\quad F(0)=0, \tag{5}$$

i.e., P is determined immediately by $f(I)$ and δ. In Eq. (4), $I_m(\delta)$ is the peak intensity of the soliton; it is defined as the minimum positive root of the equation $F(I)=\delta$. The multistability of a singular soliton is realized when the function $\delta(P)$ which is implicitly determined by Eq. (4) becomes multivalued.

It is readily shown that a "positive" Kerr-type nonlinearity (i.e., $f=\alpha I$, where $\alpha>0$), results only in a one-valued single soliton (with $\delta\propto P^2$), (see Fig. 1, curve 5), whereas "negative" Kerr-type nonlinearity ($\alpha<0$), as is well known,[11] does not provide any solitons at all. In this respect, one has to notice once again that multistable solutions discussed in this paper are still "conventional" singular solitons in the sense that they are solutions of the nonlinear Schrödinger equation (1) with a *nonvarying* amplitude profile [see the text preceding Eq. (2)] in an *infinite* (or semi-infinite) medium. On the other hand, it is known[12] that the waves in nonlinear Fabry-Perot (or ring) resonators, excited by a beam of light incident on one of the semitransparent mirrors, exhibit multistable transverse structure. In terms of an infinite medium it would correspond to propagation with periodic boundary conditions and with a driving term (i.e., incident beam) at each boundary. These multistable structures are substantially attributable to the resonant nature of the system rather than to the type of nonlinearity or to the pertinent soliton solutions. For example, bistable profiles exist[12] in resonators even with a "negative" nonlinearity, whereas the same nonlinearity does not allow existence of solitons in an infinite medium at all. In fact, none of the nonlinearities considered in Ref. 12 can produce the multistable solitons discussed in this paper. It must be noticed also that the transverse structure of the field in resonators[12] (in particular, the intensity profile) varies along the axis of propagation, in direct contrast to singular solitons.

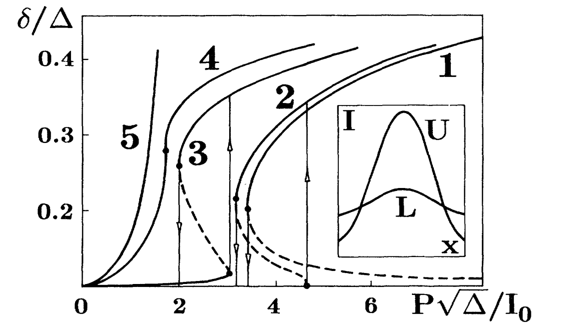

FIG. 1. Propagation constant δ vs the total power P carried by the soliton. Curves 1–5 correspond to various functions of nonlinearity: 1, step function, Eq. (8); 2, $f=a_2I^2+a_3I^3-a_4I^4$ $(a_2,a_3,a_4>0)$; 3, Eq. (14), with $a_1a_3<a_2^2S_{\rm cr}$; 4, Eq. (14), with $a_1a_3=a_2^2S_{\rm sr}$; 5, Kerr nonlinearity, $f\propto I$. The broken lines in curves 1–3 correspond to the unstable solitons. In the inset, the intensity profiles $I(x)$ are depicted of solitons that carry the same power but correspond to different branches of function δP; U, upper branch, and L, lower branch.

The absence of multistable solutions for Kerr-type nonlinearity ($f\propto I$) also holds for any other nonlinearity with $f\propto I^{\mu}$, where $\mu>0$ (but $\mu\neq 2$). The nonlinearity $f\propto I^2$ plays a special role in the two-dimensional propagation in the sense that in this case the total energy carried by any singular soliton is the same regardless of its spatial profile and propagation constant. Indeed, for $f=I^2/I_0^2$, where $I_0=$ const, the intensity profile $I(x)$ and propagation constant δ are defined[13] from Eq. (3′):

$$I(x)=I_m/\cosh(2I_mx/I_0\sqrt{3}),\quad \delta=I_m^2/3I_0^2, \tag{6}$$

where the maximum intensity of the soliton I_m is an arbitrary constant, the total power is $P_0=\pi I_0\sqrt{3}/2$. One may also show using Eq. (4) that a "positive" nonlinearity with saturation, i.e., $f=aI(1+I/I_0)^{-1}$, where a and I_0 are some positive constants, fail to produce multistable solitons. Such a nonlinear susceptibility with saturation may be attributed to various processes,[7] in particular to the interaction of appropriately tuned near-resonant light with two-level atoms (see, e.g., Butylkin, Kaplan, and Khronopulo[14]). This, again, is contrary to the resonator systems, in which this nonlinearity may cause multistability,[1] in particular, in the transverse structure.[12]

One may note from Eq. (4) that in the case of arbitrary $f(I)$, a constant δ may be viewed as a first integral ("energy") of some system with a potential $F(I)$ [Eq. (5)]. The motion of this system in some p domain can then be described by the equation

$$d^2I/dp^2+8d[F(I)]/dI=0, \qquad (7)$$

where if p is interpreted as a "time," and $P(\delta)$ is a total "period" of oscillation of the system for any given "energy" of excitation, 16δ. Indeed, the first integral of Eq. (7) with potential $8F(I)$ is given as $(dI/dp)^2=\text{const}-16F(I)$, where const may be considered as a "total energy." Therefore, the "period" P of the "oscillation" with a given amplitude I_m is $P=4\int_0^{I_m}(dI/dp)^{-1}\,dI$, which results in Eq. (4) with const $=16\delta$. Specifically, the case $f\propto I^2$ (and therefore, $F\propto I^2$) corresponds to a "linear oscillator," with the period of oscillation P independent of its "energy" δ, i.e., $dP/d\delta=0$ as suggested above.

In order to demonstrate the existence of a countable set of states for the singular soliton (with more than one state), we consider first the step nonlinearity:

$$f(I)=0,\ \text{if}\ I<I_0;\quad f=\Delta,\ \text{if}\ I>I_0, \qquad (8)$$

where I_0 and Δ are some positive constants. Substituting (8) into (4), one gets

$$P(\delta)=\frac{I_0}{\Delta^{1/2}}\frac{1}{1-\beta}\left[\frac{1}{\beta^{1/2}}+\frac{\arcsin\beta^{1/2}}{(1-\beta)^{1/2}}\right], \qquad (9)$$

$$\beta\equiv\frac{\delta}{\Delta}.$$

The function β vs P determined by (9) is a two-valued function (Fig. 1, curve 1) for any $P>P_{\rm cr1}\approx 3.44I_0/\Delta^{1/2}$ with $\beta(P_{\rm cr})\approx 0.21$. Another example is given by the nonlinearity

$$f(I)=0,\ \text{if}\ I<I_0;$$
$$f(I)=\Delta(1-I_0^2/I^2),\ \text{if}\ I>I_0. \qquad (10)$$

$f(I)$ is now a continuous function as opposed to Eq. (8). However, its derivative df/dI is still discontinuous. The total power in Eq. (4) is now

$$P=\frac{I_0}{\Delta^{1/2}}\frac{1}{1-\beta}\left[\frac{1}{\beta^{1/2}}+\frac{\arccos\beta^{1/2}}{(1-\beta)^{1/2}}\right],\quad \beta=\frac{\delta}{\Delta}, \qquad (11)$$

which essentially represents the same kind of behavior as Eq. (9), i.e., provides a two-valued soliton $\beta(P)$ for any $P>P_{\rm cr2}\approx 4.28I_0/\Delta^{1/2}$, with $\beta(P_{\rm cr})\approx 0.26$. In these cases, the nontrivial branches of the function $P(\delta)$ tend to infinity as $\delta\to 0$ and $\delta\to\Delta$ (note that the third, "trivial," branch with $\delta\equiv 0$, P arbitrary, corresponds to a nontrapped beam with $I_m<I_0$). This suggests a bistability without hysteresis and is due to the fact that the nonlinearity $f(I)$ differs from zero only for some finite $I>I_0$. The same kind of soliton bistability is exhibited by the system if either (i) $df(0)/dI<0$, but $f(I)$ becomes positive at some I, e.g., when $f=-a_1I+a_2I^2-a_3I^3$, where $a_1,a_2,a_3>0$ and $9a_1a_3<2a_2^2$, or (ii) $f(I)>0$ in the vicinity of $I=0$, but $f(I)=o(I^2)$; e.g., $f(I)=a_1I^3-a_2I^4$ $(a_1,a_2>0)$ or $f(I)=a_1I^3(1+I^3/I_0^3)^{-1}$ $(a_1,I_0>0)$. The latter nonlinearity may result from the three-photon resonant absorption of light by two-level systems with saturation.

In order to attain truly hysteretic bistable behavior [i.e., that characterized by S-shape steady-state curves (see, e.g., curves 2 and 3 in Fig. 1) which cause both "on" and "off" jumps between different branches of the curve], the function $f(I)$ must be positive at least in some range $0<I<I_1$ and have a distinct peak in its first derivative df/dI in this range. The existence of hysteretic jumps is secured if $d\delta/dP=\infty$ (or $dP/d\delta=0$) for two (or more) discrete values of P (or δ), where $dP/d\delta$ is found from (4) as

$$\frac{dP}{d\delta}=\frac{1}{2\delta}\int_0^{I_m}\left[1-2\frac{F(d^2F/dI^2)}{(dF/dI)^2}\right]\frac{dI}{[\delta-F(I)]^{1/2}}. \qquad (12)$$

The derivative $dP/d\delta$ is strongly affected by $F''(I)$ and therefore by $f'(I)$; bistability may exist if $f'(0)>0$, and if at some point $I=\tilde{I}$ we have $f''(\tilde{I})=0$ and $f'(\tilde{I})>f'(0)$. As an example of such a function, consider

$$f=a_1I+a_2I^3-a_3I^5, \qquad (13)$$

where $a_1,a_2,a_3>0$. S-shaped behavior of $\delta(P)$ (Fig. 1, curve 3) is possible if the following condition is satisfied:

$$a_1a_3/a_2^2<S_{\rm cr}=O(1), \qquad (14)$$

where $S_{\rm cr}$ is some critical quantity; a rough estimate gives $S_{\rm cr}\sim 0.1$–0.2. In the general case, the critical situation, when the curve $P(\delta)$ at some point $\delta=\delta_{\rm cr}$ has $dP/d\delta=d^2P/d\delta^2=0$ (see, e.g., Fig. 1, curve 4), corresponds to the conditions

$$dP/d\delta_{\rm cr}=0,\quad 2(d^2F/dI_{\rm cr}^2)F=(dF/dI_{\rm cr})^2, \qquad (15)$$

where $I_{\rm cr}$ is the minimal solution of the equation $\delta_{\rm cr}=F(I_{\rm cr})$, which determines both $\delta_{\rm cr}$ and the required parameters of the function $F(I)$ and therefore, $f(I)$. In the case when $f(I)=O(I^2)$ at $I=0$, the function $\delta(P)$ forms a hysteresis if $d^2f/dI^2>0$,

$d^3f/dI^3>0$, and $d^4f/dI^4<0$ at $I=0$, e.g., $f=a_2I^2+a_3I^3-a_4I^4$ $(a_2,a_3,a_4>0)$. In such a case, the lower (stable) branch of $\delta(P)$ corresponds to a nontrapped beam ($\delta=0$) (see Fig. 1, curve 2).

The stability of each of the possible solitons all of which correspond to the same total power P is an important issue. The small-perturbation analysis of the spatial stability of multistable solitons in the case of step nonlinearity (8) shows that the lower branch of curve 1, Fig. 1 corresponds to the unstable solitons and the upper corresponds to the stable ones; the trivial solution ($\delta=0$) is stable for any P. This suggests a general criterion for an arbitrary $f(I)$, and therefore $\delta(P)$: The stable solitons are those for which $d\delta/dP>0$ and vice versa (see Fig. 1, curves 1–3). In the future, it would be of considerable interest to study a "collision" of two solitons that belong to the upper and lower branches of the curve $\delta(P)$.

Bistable solitons may also exist in the case of three-dimensional propagation. Stationary self-trapping of a cylindrical beam, for instance, is governed by the "nonlinear Bessell" equation [instead of Eq. (2)]:

$$d^2u/dr^2+(1/r)(du/dr)+u[f(u^2)-\delta]=0, \quad (16)$$

where r is the radial coordinate in the plane normal to z axis. For cylindrical beams, a Kerr nonlinearity, $f\propto I$, plays the same role as $f\propto I^2$ in the two-dimensional case: For such a nonlinearity, the total power of the beam does not depend on its size or its peak intensity.[7] Therefore, in order to attain a nonhysteretic bistable soliton propagation of the kind depicted by curve 1, Fig. 1, the lowest required degree of nonlinearity at $I\rightarrow 0$ is $f\propto I^2$ [with f attaining some maximum or saturation when I increases, e.g., $f=\alpha I^2(1+I^2/I_0^2)^{-1}$]. Such a nonlinearity can originate from two-photon resonant absorption.[14] The hysteretic characteristic curve $\delta(P)$ similar to curve 2, Fig. 1, results from the nonlinearity of the form $f(I)=a_1I+a_2I^2-a_3I^3$ $(a_1,a_2,a_3>0)$, with the critical condition in the same form as Eq. (14) [but with different $S_{cr}=O(1)$].

In conclusion, the existence of multistable soliton solutions of the generalized nonlinear Schrödinger equation was demonstrated. In order for those solitons to exist, the nonlinearity must satisfy some special conditions, e.g., its dependence on the light intensity must have a range where it increases sufficiently sharply. In nonlinear optics, these solitons may manifest themselves either as singular pulses (e.g., in nonlinear fibers) or self-trapped channels (in both two- and three-dimensional cases). Bistable solitons present the ultimate case of multistable wave propagation and may find an application in dynamic (temporal) optical bistability and bistable resonator-free self-trapping of light.

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