# Renormalization of Mappings of the Two-Torus

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(Received 24 May 1985)

We show how a generalization of continued fractions can be used to develop a renormalization-group formalism to study the behavior of maps of the two-torus. Such maps may mimic the universal behavior of dynamical systems with three mutually incommensurate frequencies. Numerical evidence indicates that "chaos" may occur in maps which are invertible. While we do not see scaling at chaos onset, subcritical scaling is observed and explained by the number theory.

PACS numbers: 05.45.+b, 02.30.+g, 47.20.+m

Continued fractions and renormalization have had a key role in our understanding of transitions to chaos governed by two competing incommensurate frequencies. This approach involves the modeling of a complicated dynamical system by a simple recursion formula which one hopes retains certain universal features of the original complicated system. Maps of the circle have been used to study dynamical systems with two incommensurate frequencies,[1,2] and maps of the torus[3,4] have been investigated as models of systems where more frequencies occur.[5]

Our goal has been to understand the transition to chaos of the multifrequency dissipative torus maps by appropriately generalizing to maps of the two-torus the renormalization-group structure which has been successfully used to study the circle maps. There are three main steps to generalizing the renormalization algorithm: (1) We need to be able to approximate systematically two independent incommensurate frequencies by rational numbers with a common denominator. (2) We must use the rational approximants to construct a renormalization transformation in an appropriate function space. (3) Long cycles of a typical torus mapping must be investigated to determine if a fixed point of the renormalization group can be expected to govern the chaos transition.

(1) *Number theory.*—We first describe how to approximate simultaneously two incommensurate numbers $\sigma_x$ and $\sigma_y$ by rationals with a common denominator according to an algorithm apparently first invented by Jacobi.[6] We want to define a sequence of integers $(p_n,q_n,r_n)$ so that $\lim_{n\to\infty} p_n/r_n=\sigma_x$ and $\lim_{n\to\infty} q_n/r_n=\sigma_y$. To do this, we form the three-component vector $\boldsymbol{\rho}_0=(\sigma_x,\sigma_y,1)$. We will compute the approximating integers by a sequence of coordinate transformations. Define $P_0$ as the $3\times 3$ identity matrix. We now recursively define $\boldsymbol{\rho}_n=((\rho_n)_1, (\rho_n)_2, (\rho_n)_3)$, and $P_n$.

We assume that the rows of $P_n$ define the vectors of the $n$th coordinate axes $\hat{\mathbf{x}}_n, \hat{\mathbf{y}}_n$ and $\hat{\mathbf{z}}_n$ in the original coordinate system $\hat{\mathbf{x}}_0$, $\hat{\mathbf{y}}_0$, and $\hat{\mathbf{z}}_0$, a statement clearly true for $P_0$. Define $J_n$ as the index of the minimum of the three entries of $\boldsymbol{\rho}_n$, i.e., $(\rho_n)_{J_n} \le (\rho_n)_I$ for all $I$. We let $\mathrm{Perm}(J_n)$ be the $3\times 3$ matrix which cyclically permutes the coordinate $J_n$ of $\boldsymbol{\rho}_n$ to coordinate 3, and label the resultant intermediate permutation of $\rho$, by $\boldsymbol{\rho}_n'=\mathrm{Perm}(J_n)\boldsymbol{\rho}_n$. We now form the integers $N_n=[(\rho_n')_1/(\rho_n')_3]$ and $M_n=[(\rho_n')_2/(\rho_n')_3]$, where $[x]$ means the greatest integer less than or equal to $x$. Since we are free to multiply $\boldsymbol{\rho}_n$ by a scalar without changing relative ratios of each component, we define $(\tau_n)^{-1}=(\rho_n')_3$. We also define the matrix

$$T_{J,N,M}=\begin{bmatrix}1 & 0 & -N\\ 0 & 1 & -M\\ 0 & 0 & 1\end{bmatrix}\mathrm{Perm}(J) \tag{1}$$

and $S_{J,N,M}=(T_{J,N,M}^{-1})^{\mathrm{T}}$ where transposition is indicated. The recursion is then defined by

$$\boldsymbol{\rho}_{n+1}=\tau_n T_{J_n,N_n,M_n}\boldsymbol{\rho}_n, \quad P_{n+1}=S_{J_n,N_n,M_n}P_n. \tag{2}$$

The entries of $P_n$ are all positive and grow with $n$, while $\det(P_n)=1$ and $P_n^t\boldsymbol{\rho}_n=(\prod_{m=0}^{n-1}\tau_m)\boldsymbol{\rho}_0$. This choice for the transformation is not unique. In contrast to the ordinary continued-fraction algorithm, we can construct equally sensible transformations, for example, by choosing $J_n$ to be determined by the second-largest entry of $\boldsymbol{\rho}_n$, so that one of $M_n$ or $N_n$ is zero. This nonzero value of $M_n$ or $N_n$ could also be replaced by 1, which yields a matrix which only has a single 1 on an off-diagonal. This last construction gives the analog of the Farey approximations which have been investigated in the context of circle maps.[7] We do not believe that this indeterminacy is a serious drawback to the method; each of these candidates for the continued fraction generalization would work equally well; no known algorithm gives the "best" simultaneous approximant.

Our transformations are sequences of shears. The new basis vectors $\hat{\mathbf{x}}_n$, $\hat{\mathbf{y}}_n$, and $\hat{\mathbf{z}}_n$ in coordinates of the original lattice form the rows of $P_n$ and become longer and longer with increasing $n$. The unit cell remains with constant volume since the determinant of $T_{J,N,M}$ is one. The ray $\boldsymbol{\rho}_n$ is always in the positive octant of $R^3$ and is contained in a triangular cone defined by the basis vectors. Seen in coordinates of the original lattice, the triangular cone has smaller and smaller solid angle with increasing $n$. Each triplet of integers in each row of $P_n$ forms a simultaneous rational approximation to the ray $\rho_0$. The best of these approximants is the set

$$(p_n, q_n, r_n) = ((P_n)_{31}, (P_n)_{32}, (P_n)_{33}),$$

which we define to be the $n$th simultaneous rational approximant. The error in the approximation is of the order $\prod_{k=0}^{n-1} \tau_k^{-1}$, which converges to zero.

(2) *Renormalization transformation.*—We define the map on the two-torus by $\mathbf{f}(\mathbf{x}) = \mathbf{x} + \mathbf{\Omega} + \mathbf{g}(\mathbf{x})$ where $\mathbf{g}(\mathbf{x}+\mathbf{m}) = \mathbf{g}(\mathbf{x})$ and $f: R^2 \to R^2$ is a mapping of the two-dimensional plane $\mathbf{x} = (x,y)$ to itself, $\mathbf{m} \in Z^2$ is any pair of integers, and $\mathbf{\Omega} \in R^2$. We assume that $\mathbf{f}(\mathbf{x})$ is $C^\infty$ and invertible. The winding number is a two-component object $\boldsymbol{\rho}(\mathbf{f})$ defined by

$$\boldsymbol{\rho}(\mathbf{f}) = \lim_{n\to\infty} \frac{1}{n} \mathbf{f}^2(0) = ([\boldsymbol{\rho}(\mathbf{f})]_x, [\boldsymbol{\rho}(\mathbf{f})]_y), \quad (3)$$

where 0 is the origin. (Functions are never multiplied; the superscript on a function refers to repeated composition here and elsewhere in this article).

The curly bracket { } will denote a triplet of points in the plane and $\{\boldsymbol{\xi}\} = \{\xi_1, \xi_2, \xi_3\}$ denotes a triplet of mappings of the plane whose values are

$$\{\boldsymbol{\xi}\}(\mathbf{x}) = \{\xi_1(\mathbf{x}), \xi_2(\mathbf{x}), \xi_3(\mathbf{x})\}.$$

Here and henceforth, capital indices refer to a particular member of the triplet { }. The first triplet of the series $\{\boldsymbol{\xi}_n\} = \{\xi_{n1}, \xi_{n2}, \xi_{n3}\}$ is defined to be $\{\boldsymbol{\xi}_0\}(\mathbf{x}) = \{\mathbf{x} - \hat{\mathbf{x}}, \mathbf{x} - \hat{\mathbf{y}}, \mathbf{f}(\mathbf{x})\}$, where $\hat{\mathbf{x}} = (1,0)$ and $\hat{\mathbf{y}} = (0,1)$.

Associated with any $3\times 3$ matrix of integers $p = p_{IJ}$ we can define the operation $\otimes$ of the matrix on a function triplet $\{\xi_1, \xi_2, \xi_3\}$. The operation is similar to matrix multiplication but composition will replace addition and unless the functions commute under composition, we must be careful of the order in which we compose them. The $J$th element of $\{p \otimes \{\boldsymbol{\xi}\}\}$ will be:

$$(p \otimes \{\boldsymbol{\xi}\})_J = \xi_1^{p_{J,1}} \xi_2^{p_{J,2}} \xi_3^{p_{J,3}}. \quad (4)$$

An important special case is

$$S_{1,1,1,} \otimes \{\boldsymbol{\xi}\} = \{\xi_2, \xi_3, \xi_1\xi_2\xi_3\}.$$

In our application, it will be easy to show that all our functions $(\boldsymbol{\xi})_I$ commute so that the order of compositions is not important.

Assume that $\{\boldsymbol{\xi}_n\}$ is given. Let $\Delta\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ be the interior of the triangle defined by three arbitrary points in the plane, $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$. Let 0 be the origin. Let $\boldsymbol{\rho}_0 = ([\boldsymbol{\rho}(\mathbf{f})]_1, [\boldsymbol{\rho}(\mathbf{f})]_2, 1)$ and let $\boldsymbol{\rho}_n$ be the sequence of $\boldsymbol{\rho}_n$ defined by the Jacobi algorithm starting from this value of $\boldsymbol{\rho}_0$. Since $\mathbf{f}^{r_n}(0) - (p_n, q_n) \to 0$ for large $n$, it is natural to investigate $\xi_{01}^{p_n} \xi_{02}^{q_n} \xi_{03}^{r_n}$ which can be seen to be equivalent. We look at this multiple composition recursively by investigating the intermediate compositions $\{\boldsymbol{\xi}_{n+1}\} = S_{J_n, N_n, M_n} \otimes \{\boldsymbol{\xi}_n\}$. For all these function triplets, the origin is included in the triangle $\Delta\{\{\boldsymbol{\xi}_n\}(0)\}$. This transformation would eventually shrink the triangle $\Delta\{\{\boldsymbol{\xi}_n\}(0)\}$ to zero. To remedy this, we apply a rescale transformation with the $2\times 2$ matrix $\boldsymbol{\alpha}$ so that $\boldsymbol{\xi}_1(0) = -\hat{\mathbf{x}}$ and $\boldsymbol{\xi}_2(0) = -\hat{\mathbf{y}}$ is preserved. Thus, $\boldsymbol{\alpha}_n^{-1}$ is the matrix whose column vectors are $-(\boldsymbol{\xi}_n)_{J_n \oplus 1}(0)$ and $-(\boldsymbol{\xi}_n)_{J_n \oplus 2}(0)$, where $\oplus$ indicates addition modulo 3. Finally, we define

$$\{\boldsymbol{\xi}_{n+1}\}(\mathbf{x}) = \boldsymbol{\alpha}_n \{S_{J_n, N_n, M_n} \otimes \{\boldsymbol{\xi}_n\}\}(\boldsymbol{\alpha}_n^{-1}\mathbf{x}). \quad (5)$$

(Identical arguments or prefactors of each function inside the bracket are written outside the bracket.)

(3) *Numerical results.*—We will now focus on maps of the torus which are related to the special winding number necessary to make $N_n = M_n = 1$, $\boldsymbol{\rho}_n = \boldsymbol{\rho}_0 = \boldsymbol{\rho}$, $\tau_n = \tau$, and $J_n = 1$ for all $n$. This pair of winding numbers is the simplest analog of the golden-mean winding number for circle maps. The rescale factor $\tau$ satisfies the cubic equation $\tau^3 = \tau^2 + \tau + 1$, $\tau = 1.839\,286\,755\,21\cdots$, and $\boldsymbol{\rho} = (\tau^{-1}, \tau - 1)$. This particular choice of $\boldsymbol{\rho}$ together with our recursion formulas generates a set of rational approximants $(p_n, q_n, r_n)$. We let $\mathbf{a}$ be a set of parameters $(a_{ij})_k$ and $f_{\mathbf{a},\mathbf{\Omega}}$ the mapping

$$(f_{\mathbf{a},\mathbf{\Omega}}(\mathbf{x}))_k = x_k + \Omega_k + \sum_{n,m} \frac{(a_{nm})_k}{2\pi} \sin(2\pi(nx+my)), \quad k = 1,2, \quad (6)$$

where $k$ labels the coordinate $x$ or $y$.

For a particular set of parameters $\mathbf{a}$, we define $(\mathbf{\Omega}_n) \equiv (\mathbf{\Omega}_{nx}, \mathbf{\Omega}_{ny})$ by $f_{\mathbf{a},\mathbf{\Omega}_n}^{r_n}(0) = (p_n, q_n)$. Thus, $\mathbf{\Omega}_n$ is the parameter value which places the origin on $r_n$ cycle with winding number $\boldsymbol{\rho} = (p_n/r_n, q_n/r_n)$. We can define the $2\times 2$ scaling matrix $\boldsymbol{\delta}_n$ by $\mathbf{\Omega}_{n+1} - \mathbf{\Omega}_n = \boldsymbol{\delta}_n(\mathbf{\Omega}_n - \mathbf{\Omega}_{n-1})$ where matrix multiplication is implied. It is not difficult to show that for the case $\mathbf{a} = 0$, the two eigenvalues of $\boldsymbol{\delta}_n$ are

$$(\delta_\infty)_\pm = \tfrac{1}{2}\{-(1+\tau) \pm i(3\tau^2 + 2\tau + 3)^{1/2}\}.$$

This type of scaling is "trivial," since it follows from the number theory developed in the first part of this article. The two eigenvalues of the $2\times 2$ matrix $\boldsymbol{\alpha}$ are denoted by $(\alpha_\infty)_\pm$ and for trivial scaling are given by

$$(\alpha_\infty)_\pm = -\tfrac{1}{2}\{\tau(\tau-1) \pm i(4\tau + 1 - \tau^2)^{1/2}\}.$$

It should be noted that $|\alpha_\infty|^2 = \tau$ and $|\delta_\infty| = \tau|\alpha_\infty|$.

When $\mathbf{a} \neq 0$, we appeal to results of Arnold and Herman.[8] Let $\boldsymbol{\rho} = \boldsymbol{\rho}(f_{\Omega,\mathbf{a}})$. Loosely stated, "Part A" says that if $\boldsymbol{\rho}$ is a sufficiently mutually irrational pair of winding numbers, $f_{\mathbf{\Omega},\mathbf{a}}$ is equivalent to the shift map $f_{\boldsymbol{\rho},\mathbf{a}=0}$ via a $C^1$ change of coordinates if and only if

$\sup_{m,\mathbf{x}} ||\mathrm{Jac}(f^m_{\mathbf{\Omega},\mathbf{a}})(\mathbf{x})|| < \infty$, where $||\cdots||$ indicates matrix norm (maximum length of the image of a vector on the unit sphere) and Jac indicates the Jacobian of the $m$th iterate of the map with respect to $\mathbf{x}$. A mapping which satisfies Part A is termed "subcritical," otherwise it will be called "supercritical." Mappings which are arbitrarily close to both supercritical and subcritical mappings are termed critical, and the parameter values of these maps are said to be on the critical boundary. "Part B" states that for sufficiently small $\mathbf{a}$, the map $f_{\mathbf{\Omega},\mathbf{a}}$ is subcritical.

Our numerical results for an unsystematic sample of modest values of parameters $|(a_{ij})_k| \leq 0.5$ is consistent with Part B of Arnold's theorem. This fact has also been observed by others in numerical experiments.[3,4] All these maps are subcritical and $\boldsymbol{\delta}_n$ and $\boldsymbol{\alpha}_n$ converge to the trivial scaling form in the limit $n \to \infty$. Part B can in fact be proven[9] from our renormalization-group equation by investigating the "trivial" fixed point determined by Eq. (5) with $\mathbf{a} = 0$.

In contrast to ordinary circle maps, there is no reason to believe that the critical boundary for this problem is associated with the Jacobian of $\mathbf{f}$ becoming noninvertible at some point in the domain. Our numerical work clearly indicates that there is a large family of maps (our guess is most) which become critical at parameter values where the map is still invertible. This phenomenon is well known in area-preserving maps of the annulus.[10]

We have investigated a special critical map with parameter values $\mathbf{a} = 0$ except $(a_{01})_1 = 1$. This decoupled map could in principle show universal critical behavior if the cross-coupling terms of a nonlinear map were "irrelevant." In spite of investigating cycles to length 121 415, we were unable to discover a scaling formula for $\boldsymbol{\delta}$ and $\boldsymbol{\alpha}$ in this case and believe (based on what we know of the complicated behavior of circle maps with winding number which is not the root of a quadratic equation with integer coefficients) that critical behavior of decoupled maps is not governed by a simple fixed point. In spite of the absence of a fixed point, we can discover whether or not the critical behavior is stable against small perturbations coupling the $\mathbf{x}$ and $\mathbf{y}$ variables. If we use standard scaling arguments, this can be done by computing

$$F_n[\kappa](\mathbf{x}) = \frac{d}{d\kappa} \mathrm{Jac}(f^{r_n}_{\mathbf{\Omega}_\infty, a})(\mathbf{x}), \qquad (7)$$

where $\kappa$ indicates any one of the parameters $(a_{ij})_k$ or $\mathbf{\Omega}$, or a small parameter multiplying any other function one might want to add to the form of $f_{\mathbf{a},\mathbf{\Omega}}$. One then checks if any of these terms which do not vanish as $n \to \infty$ are in the subspace spanned by $F_n[\Omega_x]$, $F_n[\Omega_y]$, and $F_n[(a_{01})_1]$. On the basis of this calculation, we conclude that the critical decoupled torus map is unstable against cross-coupling terms of the type $(a_{1,-1})_k$ and hence that decoupled circle maps are not physically relevant for generic torus maps.

As a paradigm for the strongly coupled circle maps, we investigated the one-parameter family of circle maps $\mathbf{a} = 0$ except $a \equiv 2(a_{01})_1 = \frac{1}{2}(a_{10})_2$. In this case $f_{\mathbf{a},\mathbf{\Omega}}$ becomes noninvertible when $a = 1$. We were unable to find a precise value of the critical boundary although in this case, too, we checked cycles up to length 121 415. We have, however, strong numerical evidence based on Part A of Herman's theorem indicating that the critical point $a_c$ is below $a = 1$. Conservative bounds are $0.96 < a_c < 0.97$. The bound on $a_c$ becomes apparent only for cycles of length greater than 19 513; it converges extremely slowly in $n$, although at fixed $r_n \geq 19\,513$, the value of the boundary appears much better defined than the uncertainty in the bound of $a_c$ would suggest. We were unable to discover a scaling form for either $\boldsymbol{\delta}_\infty$ or $\boldsymbol{\alpha}_\infty$. (Lack of scaling for mappings whose Jacobians *vanish* has previously been observed by Guckenheimer, Hu, and Rudnick.[4])

The image of a $100 \times 100$ grid on the unit square under 149 iterates of this torus map with $a = 0.97$ and $\mathbf{\Omega}$ chosen so that the origin is a member of a stable 149-cycle is shown in Fig. 1. There is clear evidence of the appearance of a singularity of the Jacobian of the iterated map. We claim that this parameter value is an upper bound for the critical value of $a$.

Any one of several reasons could explain why we have not observed simple scaling. We chose somewhat arbitrarily to include the origin in the cycles for which we calculate $\Omega_n$. However, in this problem no criterion singles out the origin as the proper point about which to scale. Simple scaling about a

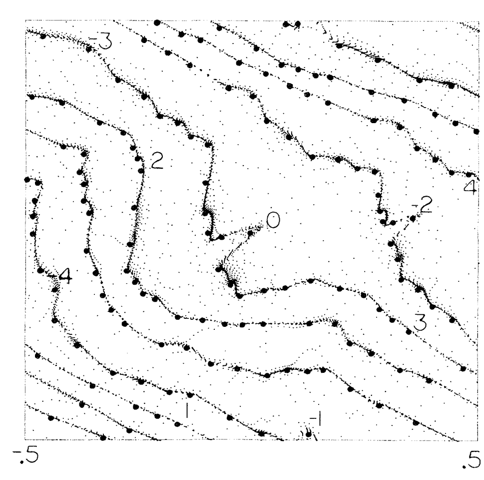


FIG. 1. The image of a $100 \times 100$ grid of points filling the unit square is show for a supercritical map corresponding to $a = 0.97$ under 149 iterates of the map. The elements of the 149-cycle which includes the origin are shown as particularly dark dots. A few images and preimages of the origin have been labeled.

renormalization-group fixed point can only be discovered if we expand about special points of the domain; expanding about any other point will destroy simple scaling. It is also possible that critical behavior is not governed by a simple renormalization-group fixed point or that a fixed point exists, but our map shows very slow crossover to that fixed point and/or leading complex eigenvalues. Finally, it is possible that the fixed point may have associated homoclinic points, leading to a "chaotic" renormalization-group trajectory.[4] We are trying to determine which of these scenarios occurs. Lack of knowledge of either an exact critical value of $a_c$ or the proper point about which to scale together with the extremely long cycles which appear to be necessary has made further progress very slow.

We have shown how to construct a renormalization-group theory for maps of the $n$-tori, and have numerical evidence that the chaos transition occurs before the maps become critical. Subcritical scaling is determined by the number theory we develop. We believe that these statements are "generic" for $n$-torus mappings when $n \geqslant 2$. For experimentalists who want to probe scaling at the transition to chaos the situation is discouraging; the necessity of investigating extremely long cycles makes observation of any simple scaling behavior which might exist appear to be hopelessly beyond the realm of any realistic experiment.

We particularly appreciate discussions we have had with Joe Rudnick and John Guckenheimer who communicated many of their unpublished results obtained with Bambi Hu. One of us (S.O.) would also like to thank Predrag Cvitanovic for many stimulating discussions. This research was supported by the National Science Foundation under Contract No. DMR-8216718 at the Laboratory for the Research in the Structure of Matter at the University of Pennsylvania and by Contract No. DE/AC02-83ER13044 at the Laboratory for Atomic and Solid State Physics at Cornell.

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