Quantum Oscillations in Normal-Metal Networks

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A general formalism is outlined for the calculation of the transport coefficients of a normal-metal network in the weak-localization regime. Simple circuits such as loops and ladders are used to illustrate our approach. A closed expression for the magnetoresistance of an infinite regular network is derived. We find that, in contrast with superconducting networks, no fine structure due to interference effects between adjacent loops is expected. Our results agree very well with the recently observed oscillations in normal-metal networks.

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Since the prediction[1] of a Bohm-Aharonov–type effect in disordered metals, with half-quantum flux $\phi_0 = hc/2e$, only two groups were able to observe clearly this effect in the following new multiconnected geometries: regular networks[2] and ladders.[3] The original experiment, performed on a hollow cylinder, has been repeated by several groups.[4] The magnetoresistance (MR) oscillations observed in these experiments are actually the manifestation of a specific and new phenomenon in disordered materials. The physical explanation in terms of coherent backscattering has been advanced[5] in the case of electronic transport. However, the interference effect, obtained originally through an explicit diagram calculation, is really a very general phenomenon in systems with quenched disorder. Indeed, the basic origin must be traced to the amplification of the backscattering during the propagation of waves in randomly inhomogeneous media, where multiple scattering dominates. As long as $\lambda << l$ (λ is the wave length and l is the mean free path), the first interference corrections to the wave-field energy-transport equation are controlled by the so-called fan diagrams. This is actually the case in the weak-localization regime. The presence of a magnetic field, which couples to the phase of a wave function, is therefore the most direct method to reveal the interference effects.

Given the fundamental aspect of the interference phenomena in disordered metals, it is natural to look at the corresponding corrections in new geometries, like networks, where the recent experiments were performed. The magnitude of the MR oscillations has been calculated only for the hollow-cylinder geometry, and there is no equivalent expression for the general situation. In addition to the relevance of such a calculation for the experimental investigations, there are at least two additional motivations for our study. Firstly, how is the amplitude of the MR oscillation influenced by the experimental setup? Secondly, are there new features of the MR curve in the case of an extended network? Actually, such effects will be produced by interferences between adjacent loops in the network. For instance, in superconducting networks, such effects were predicted and observed[6] on the fine structure of the upper critical line. Is there a counterpart in the case of normal networks?

In this Letter, we report on a general formalism for the calculation of transport coefficients for a normal-metal network of arbitrary shape. Our formulation, illustrated below on some examples, permits us to answer the above questions and provides explicit expressions for the MR oscillations for an arbitrary network. In the following, we will limit our exposition to localization corrections in the weak-localization regime. Note, however, that corrections due to electron-electron interaction can also be calculated in the framework of the present formulation. A more detailed exposition will be given elsewhere.[7]

The localization correction to the conductivity in the weak-localization regime ($k_F l >> 1$) is given in general by the following expression[8]:

$$\Delta\sigma(\mathbf{r}) = -(2/\pi\nu)\sigma_0 C(\mathbf{r},\mathbf{r}), \qquad (1)$$

where σ_0 is the bulk conductivity of the sample, given by Drude's formula, and ν is the density of states at the Fermi level. The equation for the Cooperon[9] $C(\mathbf{r},\mathbf{r}')$ in the presence of a magnetic field (vector potential $\mathbf{A}$) is

$$\{[-i\nabla_{\mathbf{r}} - (2\pi/\phi_0)\mathbf{A}(\mathbf{r})]^2 + L_\phi^{-2}\}C(\mathbf{r},\mathbf{r}') = (1/\hbar D)\delta(\mathbf{r}-\mathbf{r}'). \qquad (2)$$

Here D denotes the electron diffusion coefficient and $L_\varphi = (D\tau_\phi)^{1/2}$ is the length over which dephasing of the electron wave function results from inelastic processes or of spin-spin scattering from paramagnetic centers. Equation (2) must be supplied by a boundary condition on the surface of a given sample. In the following, we shall confine ourselves to the free boundary conditions. It is important to notice that Eqs. (1) and (2) correspond actually to a self-averaged theory, where all traces of randomness are summarized in L_φ.

As can be seen from Eq. (1), the correction to the conductivity depends on the coordinates. However, since $\Delta\sigma(\mathbf{r})$ is a small correction to the total conductivity, the total correction to the measured resistance, for instance, is obtained by integrating $\Delta\sigma(\mathbf{r})$ over the volume of the system.

In order to calculate $\Delta\sigma(\mathbf{r})$ in a multiply connected geometry, Eq. (2) leads to formidable calculations, already in the case of simple circuits such as the hollow cylinder or single rings.[1] The situation becomes very simple in the thin-wire approximation used here. Indeed, we shall investigate networks made of thin wires, having a thickness much smaller than L_φ. This corresponds to wires of effective dimensionality one, and this condition is actually fulfilled in the experiments done on networks.[2] The finite width of wires can, however, be taken into account, and this results in a renormalization of L_φ which becomes a function of the magnetic field (see below). This formalism breaks down in very small systems[10] where there is a lack of self-averaging. If we assume that $C(\mathbf{r}',\alpha)$ is known at two adjacent nodes of the network, it is straightforward to deduce $C(\mathbf{r}',\mathbf{r})$ for any point $\mathbf{r}$ on the strand $(\alpha\beta)$. This remark permits us to write down a set of Kirchoff-type equations, leading to the following network equations:

$$\sum_\beta{}' \coth\left(\frac{l_{\alpha\beta}}{L_\varphi}\right) C(\alpha,\mathbf{r}') - \sum_\beta{}' \left[\frac{e^{-i\gamma_{\alpha\beta}}}{\sinh(l_{\alpha\beta}/L_\varphi)}\right] C(\beta,\mathbf{r}') = \frac{L_\varphi}{\hbar DS}\delta_{\alpha,\mathbf{r}'}. \tag{3}$$

In this basic equation, $l_{\alpha\beta}$ refers to the length of the strand $(\alpha\beta)$ between nodes α and β of the network, and $\gamma_{\alpha\beta} = (2\pi/\phi_0)\int_\alpha^\beta \mathbf{A}\cdot d\mathbf{l}$ denotes the circulation of the vector potential $\mathbf{A}$ along this strand. The sums in Eq. (3) are taken over nodes β connected to node α, and S is the cross-sectional area of the wires. In this equation, the point $\mathbf{r}'$, where the correction to the conductivity is calculated, acts as an additional node. This remark, as well as other observations,[7] shows the basic differences between Eq. (3) and the similar one derived for superconducting networks.[11] Starting from Eq. (3), one can check that we recover the known[1] result for a single loop in a normal magnetic field,

$$\frac{\Delta R}{R} = \frac{\frac{1}{2}\kappa\sinh(L/L_\varphi)}{\cosh(L/L_\varphi) - \cos(2\pi\phi/\phi_0)}, \tag{4}$$

where $\kappa = (2e^2/\pi\hbar\sigma_0)L_\varphi/S$. Here L denotes the length of the loop, and ϕ the magnetic flux through its surface.

Note that Eq. (3) can also be used in the case of a thin wire with dangling side branches.[7] In this geometry, the local character of $\Delta\sigma(\mathbf{r})$ is well illustrated where a nonmonotonic behavior of $\Delta\sigma(\mathbf{r})$ is obtained.

In general, for a network of arbitrary shape, a compact expression for $\Delta R/R$ can be derived, if we take into account the linearity of Eq. (3). For this, we shall introduce the following $N\times N$ Hermitean matrix M, where N is the number of nodes in the network:

$$M_{\alpha\alpha} = \sum_\beta{}' \coth\left(\frac{l_{\alpha\beta}}{L_\varphi}\right) - 2\sum_{\text{loops}} \frac{\cos(2\pi\phi_s)/\phi_0}{\sinh(l_s/L_\varphi)}, \tag{5a}$$

$$M_{\alpha\beta} = -\frac{e^{-i\gamma_{\alpha\beta}}}{\sinh(l_{\alpha\beta}/L_\varphi)}, \quad \alpha\neq\beta. \tag{5b}$$

In Eq. (5a), the first sum is taken over nodes β connected to node α by a strand of length $l_{\alpha\beta}$. The second sum is taken over elementary loops of length l_s containing the node α and defining a magnetic flux ϕ_s.

With use of the matrix M, the correction to the total resistance can be written for a network of arbitrary shape. In particular, for a regular network, such as the square lattice (see below), made of identical strands with $l_{\alpha\beta}=a$, the correction of the total resistance is given by[7] $(\eta = a/L_\varphi)$

$$\frac{\Delta R}{R} = \frac{\kappa}{2}\left[\left(1-\frac{2}{z}\right)\frac{\eta\cosh\eta - \sinh\eta}{\eta\sinh\eta} + \frac{2}{N}\sum_{i=1}^{N}\lambda_i^{-1}\right]. \tag{6}$$

Here z denotes the coordination number of the lattice, and $\lambda_i \geqslant 0$ denotes an eigenvalue of the matrix M.

In the following we shall illustrate the above equations in three particular cases.

(i) *Ladders.*—We have studied different networks where two or many loops are connected through arms or contact points. In general, the presence of arms damps out the MR oscillations. Furthermore, for a set of two loops, with a common node or a common edge, regular oscillations with a periodic behavior of MR is obtained for rational values of the flux ratio ϕ_1/ϕ_2. However, for two identical square loops with a common edge, no secondary maxima (i.e., at $\phi_{1,2}/\phi_0 = \frac{1}{2}$) are obtained, in contrast with an intuitive expectation. This behavior is well illustrated in Fig. 1 for a simple strip (ladder) made of identical square loops of side a each. The absence of new features of the MR persists on a multistrip of arbitrary width.

(ii) *Infinite square lattice.*—This case can be studied either directly or as a limit $(M=\infty)$ of a multistrip of width M. Both approaches lead to the same results. Let us describe the direct approach based on Eq. (6), where $z=4$. For a magnetic field H, normal to the planar network, one can take for convenience the Landau gauge, $A_x = -Hy$, $A_y = 0$, and use the translation symmetry in direction x. The eigenvalue problem can be written as a Harper's equation[12]:

$$\epsilon\psi_m = \psi_{m-1} + \psi_{m+1} + 2\cos(m\gamma + \theta)\psi_m, \qquad (7)$$

where $\epsilon = 4\cosh\eta - \lambda\sinh\eta$, $\eta = a/L_\varphi$, $\gamma = 2\pi\phi/\phi_0$, and θ denotes a Floquet factor $(0 \leqslant \theta \leqslant 2\pi)$. Here, a refers to the side of elementary plaquettes and $\phi = Ha^2$. Equation (7) can be solved for rational $\phi/\phi_0 = p/q$ (p,q integers and prime to each other). With use of Bloch's theorem, the secular equation, giving the eigenvalues ϵ, can be cast as a polynomial equation: $P_{p,q}(\epsilon) - W = 0$, where $P_{p,q}(\epsilon)$ is a polynomial of degree q in ϵ, and W denotes a parameter in the interval $[-4, +4]$. For more details, we direct the reader to Refs. 7 and 12, and Wannier, Obermair, and Ray.[13] Here we quote just the final result for the localization correction to the total resistance

$$\frac{\Delta R}{R} = \frac{\kappa}{4}\left\{\frac{\eta\cosh\eta - \sinh\eta}{\eta\cosh\eta} + \frac{8\sinh\eta}{\pi q}\frac{P'_{p,q}(4\cosh\eta)}{P_{p,q}(4\cosh\eta)}K\left[\frac{4}{P_{p,q}(4\cosh\eta)}\right]\right\}. \qquad (8)$$

Here $P'(x)$ denotes the derivative of the polynomial $P_{p,q}$ taken at $x = 4\cosh\eta$ and $K(x)$ refers to the elliptic integral of first kind,

$$K(x) = \int_0^{\pi/2} dt/(1 - x^2\sin^2 t)^{1/2}.$$

In Fig. 1, we have shown Eq. (8) as a function of the reduced flux p/q for values of q up to $q = 50$. As can be seen, $\Delta R/R$ exhibits actually a periodicity at integer values of ϕ/ϕ_0 only, as was anticipated before. The whole curve is a smooth one and indeed analytical.[7] In fact, despite the rich structure of the spectrum[11] associated with Eq. (7), $\Delta R/R$ is given by a regularizing sum [Eq. (6)] over the subbands of this spectrum, weighted by the density of states, where logarithmic singularities occur. However, Eq. (8) is analytic, because $4\cosh\eta$ lies outside the spectrum $|\epsilon| \leqslant 4$ of Eq. (7). This result is to be compared with the singular behavior of the edge of this spectrum, measured in superconducting networks.[6]

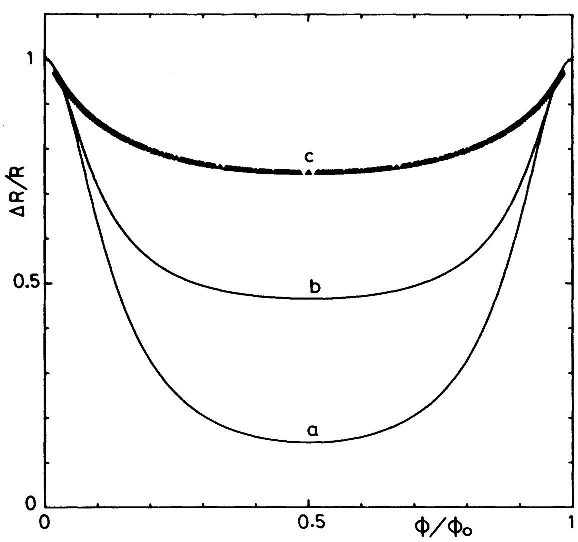

FIG. 1. $\Delta R/R$ as a function of the reduced flux ϕ/ϕ_0 shown for three networks $(a/L = 0.2)$: curve a, single square loop of perimeter $L = 4a$ [Eq. (4)]; curve b, simple ladder made of identical square loops; and curve c, infinite regular network made of identical square loops [Eq. (8)]. For convenience, $\Delta R/R$ has been normalized to its value at zero field in each case. Triangles, corresponding to case c, are calculated for rational $\phi/\phi_0 = p/q$, $p \leqslant q$, and $q \leqslant 50$, according to Eq. (8) (see text).

It is interesting to look at various limits of Eq. (8). Let us consider first the case of zero magnetic field. For $\eta \to 0$, i.e., $a \ll L_\varphi$, one gets $\Delta R/R = (\kappa/2\pi)(a/L_\varphi)\ln(L_\varphi/a)$. This result reproduces the bulk expression[8] $\Delta R = (e^2/\pi^2\hbar)R_\square^2 \ln(L_\varphi/l)$ with a, instead of l, as a cutoff at short length scales. Note that our formalism makes sense only for $a > l$, and it is natural to recover this result in the continuum limit. In the limit of small but finite magnetic field, the continuum results[8] for the magnetoresistance are also recovered: $\Delta R(H) \sim H^2$ at $\phi/\phi_0 \ll \eta^2$ and $\Delta R(H) \sim \ln(H/\eta^2)$ at $\phi/\phi_0 \gg \eta^2$.

Let us conclude with two comments relative to Fig. 1, where $\Delta R/R$ oscillation is shown for three geometries. Firstly, the absence of new interference effects (e.g., secondary maxima at rational ϕ/ϕ_0) in the ladder, as well as in the infinite network, comes from the expression of $\Delta R/R$ itself. It is actually a whole integral information over the spectrum of matrix M, and the period of MR oscillation is the same. Secondly, the amplitude of the oscillations is strongly influenced by the geometry of the considered network. This is clear at $\eta \ll 1$, where for noninteger values of ϕ/ϕ_0, $\Delta R/R \sim \eta$ in both cases. However, for integer ϕ/ϕ_0, $\Delta R/R$ is of order η^{-1}, η^0, and $\eta\ln(1/\eta)$, respectively, in the single loop, ladder, and infinite network.

(iii) *Honeycomb lattice.*—The same calculations have been performed[7] on honeycomb lattices where MR oscillations have been measured.[2] In order to make a close contact with experiments, the width of the wires must be taken into account. This results in a renormalization of the reduced factor $\eta = a/L$:

$$\eta^2(H) = \eta^2(H=0) + \tfrac{4}{81}\pi^2(\phi/\phi_0)^2(w/a)^2,$$

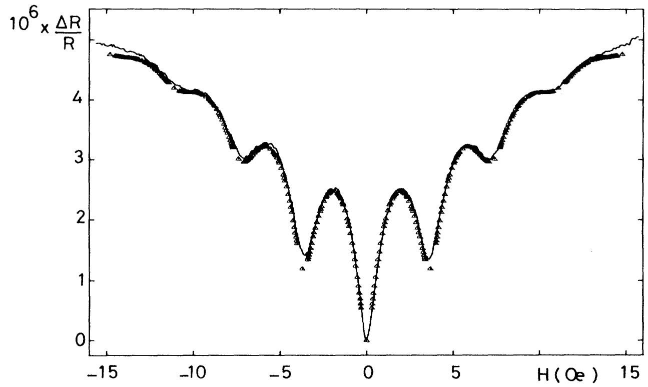

FIG. 2. Quantitative comparison between the theoretical results (triangles) and experimental data (solid line), for Cu at $T = 133$ mK, taken from Ref. 2. The hexagonal elementary cells (side $a = 1.5$ μm) are made of wires of width 0.42 μm. In this fit, we have $L_\varphi = 5.36$ and $L_{\mathrm{s.o.}} = 3.12$ μm, respectively ($L_{\mathrm{s.o.}}$ is the spin-orbit length).

where ϕ denotes the magnetic flux through an elementary hexagonal cell (side a) made of wires of width w. This low-field approximation breaks down at $Haw \gtrsim \phi_0$. The renormalization of η becomes important at large H and is actually responsible for the damping of oscillations. Our results are illustrated in Fig. 2 where spin-orbit scattering has been taken into account. Clearly, there is a fairly good agreement between theory and experimental.

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