Measurement of the Lyapunov Spectrum from a Chaotic Time Series

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The exponential divergence or convergence of nearby trajectories (Lyapunov exponents) is conceptually the most basic indicator of deterministic chaos. We propose a new method to determine the spectrum of several Lyapunov exponents (including positive, zero, and even negative ones) from the observed time series of a single variable. We have applied the method to various known model systems and also to the Rayleigh-Bénard experiment, and have elucidated the dependence of the Lyapunov exponents on the Rayleigh number.

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The deterministic unpredictable behavior of nonlinear dynamical systems has become a very interesting subject in many fields of science. Especially fluid systems, for example the Rayleigh-Bénard convection system, provide one of the most challenging problems to physicists, because they can exhibit not only chaotic behavior with a few degrees of freedom accompanied by some bifurcations, but also spatio-temporal irregular behavior (turbulence). In the case of low-order chaos, how do the physical quantities of chaos grow with increasing values of nonequilibrium parameters or with the increasing number of degrees of freedom? In the case of turbulence, how is it characterized in terms of the theory of a dynamical system?

To answer these questions, detailed characterization of the properties of the irregular behavior is required. Therefore, it is strongly desirable to develop a powerful method which is applicable to many-dimensional chaos to extract physical quantities from experimentally obtained irregular signals. The basic quantities to characterize chaotic behavior are the exponential divergence of nearby orbits (positive Lyapunov exponents[1]), positive finite Kolmogorov entropy,[2] and a noninteger dimension of the attractor.[3–5] These quantities are invariant under smooth transformation of coordinates. There are several relations among these quantities, and if the Lyapunov spectrum can be determined, the rest can be estimated as equalities or upper or lower bounds.[6–8]

To implement this procedure in an experiment, we have to calculate a set of Lyapunov exponents, positive, zero, and negative ones, simultaneously. In recent reports, two methods which can determine Lyapunov exponents from a time series have been reported.[9,10] However, these methods have some limitations; for example, the obtainable exponents are restricted to be nonnegative, and the number of obtainable exponents is one or two. In this paper, we present a new method by which one can determine a set of several Lyapunov exponents, positive, zero, and even negative, from the observed time series of a single variable. This method was tested on various known systems. We have also applied this method to the data of the Rayleigh-Bénard experiment and elucidated the dependence of the Lyapunov exponents on the Rayleigh number for the case of intermittent chaos.

Let us consider an observed trajectory $\mathbf{x}(t)$, which can be considered as a solution of a certain dynamical system:

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}), \tag{1}$$

defined in a d-dimensional phase space. On the other hand, the evolution of a tangent vector $\boldsymbol{\xi}$ in a tangent space at $\mathbf{x}(t)$ is represented by linearizing Eq. (1),

$$\dot{\boldsymbol{\xi}} = \mathbf{T}(\mathbf{x}(t)) \cdot \boldsymbol{\xi}, \tag{2}$$

where $\mathbf{T} = D\mathbf{F} = \partial \mathbf{F}/\partial \mathbf{x}$ is the Jacobian matrix of $\mathbf{F}$. The solution of the linear nonautonomous Eq. (2) can be obtained as

$$\boldsymbol{\xi}(t) = A^t \boldsymbol{\xi}(0), \tag{3}$$

where A^t is the linear operator which maps tangent vector $\boldsymbol{\xi}(0)$ to $\boldsymbol{\xi}(t)$. The mean exponential rate of divergence of the tangent vector $\boldsymbol{\xi}$ is defined as follows:

$$\lambda(\mathbf{x}(0), \boldsymbol{\xi}(0)) = \lim_{t \to \infty} \frac{1}{t} \ln \frac{\| \boldsymbol{\xi}(t) \|}{\| \boldsymbol{\xi}(0) \|}, \tag{4}$$

where $\| \ldots \|$ denotes a norm with respect to some Riemannian metric. Furthermore, there is a d-dimensional basis $\{\mathbf{e}_i\}$ of $\boldsymbol{\xi}(0)$, for which λ takes values $\lambda_i(\mathbf{x}(0)) = \lambda(\mathbf{x}(0), \mathbf{e}_i)$. These can be ordered by their magnitudes $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_d$, and are the spectrum of Lyapunov characteristic exponents. These exponents are independent of $\mathbf{x}(0)$ if the system is ergodic.[11]

We often have no knowledge of the nonlinear equations of the system which produces the observed time series. And even if we know the equations of motion, such as the Navier-Stokes equations for fluid systems, it is a hard task to derive the mode-truncated equations with finite degrees of freedom from partial differential equations (which is the infinite-dimensional system) and reproduce the same phenomena as the experiment from them.[12] However, there is a possibility of estimating a linearized flow map A^t of tangent space from the observed data.

Let $\{\mathbf{x}_j\}$ $(j=1,2,\ldots)$ denote a time series of some physical quantity measured at the discrete time interval Δt, i.e., $\mathbf{x}_j \equiv \mathbf{x}(t_0+(j-1)\Delta t)$. Consider a small ball of radius ϵ centered at the orbital point $\mathbf{x}_j$, and find any set of points $\{\mathbf{x}_{k_i}\}$ $(i=1,2,\ldots,N)$ included in this ball, i.e.,

$$\{\mathbf{y}^i\} = \{\mathbf{x}_{k_i}-\mathbf{x}_j \mid \|\mathbf{x}_{k_i}-\mathbf{x}_j\| \leq \epsilon\}, \tag{5}$$

where $\mathbf{y}_i$ is the displacement vector between $\mathbf{x}_{k_i}$ and $\mathbf{x}_j$. We used a usual Euclidean norm defined as follows: $\|\mathbf{w}\| \equiv (w_1^2+w_2^2+\ldots+w_d^2)^{1/2}$ for some vector $\mathbf{w}=(w_1,w_2,\ldots,w_d)$. After the evolution of a time interval $\tau=m\Delta t$, the orbital point $\mathbf{x}_j$ will proceed to $\mathbf{x}_{j+m}$ and neighboring points $\{\mathbf{x}_{k_i}\}$ to $\{\mathbf{x}_{k_i+m}\}$. The displacement vector $\mathbf{y}^i=\mathbf{x}_{k_i}-\mathbf{x}_j$ is thereby mapped to

$$\{\mathbf{z}^i\} = \{\mathbf{x}_{k_i+m}-\mathbf{x}_{j+m} \mid \|\mathbf{x}_{k_i}-\mathbf{x}_j\| \leq \epsilon\}. \tag{6}$$

If the radius ϵ is small enough for the displacement vectors $\{\mathbf{y}^i\}$ and $\{\mathbf{z}^i\}$ to be regarded as good approximation of tangent vectors in the tangent space, evolution of $\mathbf{y}^i$ to $\mathbf{z}^i$ can be represented by some matrix A_j, as

$$\mathbf{z}^i = A_j\mathbf{y}^i. \tag{7}$$

The matrix A_j is an approximation of the flow map A^τ at $\mathbf{x}_j$ in Eq. (3). Let us proceed to the optimal estimation of the linearized flow map A_j from the data sets $\{\mathbf{y}^i\}$ and $\{\mathbf{z}^i\}$. A plausible procedure for optimal estimation is the least-square-error algorithm, which minimizes the average of the squared error norm between $\mathbf{z}^i$ and $A_j\mathbf{y}^i$ with respect to all components of the matrix A_j as follows:

$$\min_{A_j} S = \min_{A_j} \frac{1}{N}\sum_{i=1}^{N} \|\mathbf{z}^i - A_j\mathbf{y}^i\|^2. \tag{8}$$

Denoting the (k,l) component of matrix A_j by $a_{kl}(j)$ and applying condition (8), one obtains $d\times d$ equations tions to solve, $\partial S/\partial a_{kl}(j)=0$. One will easily obtain the following expression for A_j:

$$A_jV=C,\quad (V)_{kl}=\frac{1}{N}\sum_{i=1}^{N} y^{ik}y^{il},\quad (C)_{kl}=\frac{1}{N}\sum_{i=1}^{N} Z^{ik}y^{il}, \tag{9}$$

where V and C are $d\times d$ matrices, called covariance matrices, and y^{ik} and z^{ik} are the k components of vectors $\mathbf{y}^i$ and $\mathbf{z}^i$, respectively. If $N \geq d$ and there is no degeneracy, Eq. (9) has a solution for $a_{kl}(j)$.

Now that we have the variational equation in the tangent space along the experimentally obtained orbit, the Lyapunov exponents can be computed as

$$\lambda_i = \lim_{n\to\infty} \frac{1}{n\tau}\sum_{j=1}^{n} \ln \|A_j\mathbf{e}_i^j\|, \tag{10}$$

for $i=1,2,\ldots,d$, where A_j is the solution of Eq. (9), and $\{\mathbf{e}_i^j\}$ $(i=1,2,\ldots,d)$ is a set of basis vectors of the tangent space at $\mathbf{x}_j$. In the numerical procedure, choose an arbitrary set $\{\mathbf{e}_i^j\}$. Operate with the matrix A_j on $\{\mathbf{e}_i^j\}$, and renormalize $A_j\mathbf{e}_i^j$ to have length 1. Using the Gram-Schmidt procedure, maintain mutual orthogonality of the basis (see, e.g., Shimada and Nagashima[13]). Repeat this procedure for n iterations and compute Eq. (10). The advantage of the present method is now clear, since we can deal with arbitrary vectors in a tangent space and trace the evolution of these vectors. In this method, these vectors are not restricted to observed data points, in contrast with the conventional methods.[9,10] The feature allows us to compute all exponents to good accuracy with great ease.

Our method was tested on various chaotic dynamical systems to see if it can be used for characterizing an experimental system such as the Rayleigh-Bénard system. For this purpose, a single variable of the model system (e.g., the x coordinate) was treated as experimental data, except the Hénon map, and then a d-dimensional orbit was reconstructed by use of delay coordinates,[14] i.e.,

$$\mathbf{x}_i = \{x(i\tau),\ldots,x(i\tau+(d-1)t_d)\},$$

where t_d is the delay time. We have not searched all the points contained in the ϵ ball, because it is time consuming, but we set an upper limit to the number N of points included. We proceed as follows: First we choose an orbital point $\mathbf{x}_j$ and search the points included in the ϵ ball from the beginning of the data file $\{\mathbf{x}_j\}$ $(j=1,\ldots,M)$. When the number of points found in the ϵ ball exceeds the upper limit we stop the searching and proceed to solve Eq. (9). In the case that the number does not exceed the upper limit, though the data file is exhausted, if the number satisfies the condition $N\geq d$ we proceed to solve Eq. (9), but if the condition is not satisfied we abandon this point $\mathbf{x}_j$ and go to the next point $\mathbf{x}_{j+m}$. For the value of the upper limit of N we chose 20 in this paper. It was confirmed that lower values of the limit, e.g., $d\leq N\leq 5$ for the system with $d=3$, gave similar results.

Figure 1 shows an example of the convergence of Eqs. (10) for the Lorenz equations.[15] In Fig. 1 we plot λ_i vs $n\tau$ for $\tau=0.05$, $d=3$, $t_d=0.13$, $\epsilon=(1.5\%$ of the horizontal extent), and $d\leq N\leq 20$. Convergence of λ_i is attained in an early stage of iterations. The variation of the spectrum obtained from our algorithm when the parameters ϵ, τ, and td are changed is within the values shown in Table I. It means that they are very insensitive to the choice of these parameters. The agreement with the known values is very good for λ_1 and λ_2. The next example is the Mackey-Glass delay differential equation with a delay constant $T=30$.[16,17] The estimated spectrum is in good agree-

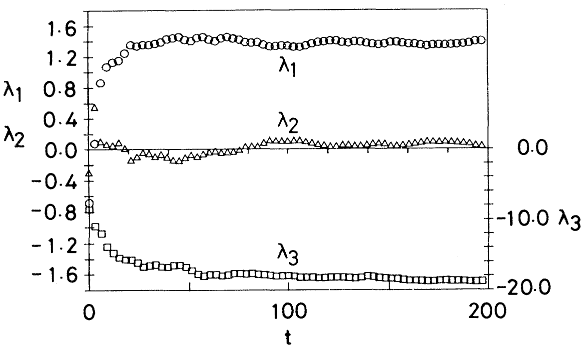

FIG. 1. Stability of the Lyapunov spectrum of the Lorenz system for a special choice of parameters. The values of the parameters are listed in Table I, except $\tau = 0.05$, $\epsilon/L = 0.015$, and $n = 2000$.

ment with the numerical results, again. It is worth noting that the method gives not only two positive exponents, but also zero and two negative ones. For the Hénon map,[18] with $a = 1.4$, $b = 0.3$, we obtained two exponents (0.408,−1.58). The error is within 1–2% for each exponent.

The possibility of measuring negative exponents depends on their magnitudes and the signal-to-noise ratio of the data. If contraction of the phase-space volume is so strong that the thickness of the attractor becomes smaller than the resolution of observation, one never has information concerning the direction of contraction. In such a case experimental noise gives spurious zero exponents for those directions, because the perturbation caused by noise is never contracting nor expanding on average. An example of the above situation is the Roessler system.[19] As shown in Table I, $|\lambda_3|$ is almost a hundred times greater than λ_1. As a consequence estimation of λ_3 was unsuccessful in this case. Detailed analysis of the method and the effect of noise will be published elsewhere.[20]

Next we make a rough estimate of the amount of data required by our method. For the sake of simplicity, consider a strange attractor with the information dimension D,[4] and let L be the horizontal extent of

TABLE I. Lyapunov spectrum, estimated fractal dimension D_{KY}, and Kolmogorov entropy for various systems. δt is the time step of numerical integration, n is the number of averaging in Eq. (10), and d is the embedding dimension. The error bars for our algorithm are calculated from several runs with different parameters ϵ, τ, and t_d around the values listed below.

System	Lyapunov exponents (numerical results)	Lyapunov exponents (our algorithm)	
Henon map	0.417±0.006 (Ref.18)	0.408±0.003	(M=10000,n=1000
(a=1.4,b=0.3)	−1.58 ±0.006	−1.58 ±0.02	,ε/L=0.01)
	K=0.417, D_{KY}=1.26	K=0.408, D_{KY}=1.26	
Lorentz equations	1.37 (Ref.13)	1.37±0.08	(M=4·10^4,n=1000,d=3
(R=40,σ=16,b=4,δt=0.01)	0.00	−0.02±0.09	,τ=0.1,ε/L=0.02)
	−22.37	−15.2 ±2.1	
	K=1.37, D_{KY}=2.06	K=1.37, D_{KY}=2.09	
Roessler equations	0.069 ±0.003	0.073 ±0.004	(M=4·10^4,n=1000,d=3
(a=b=0.2,c=5.7,δt=0.12)	−0.0002±0.0002	0.0003±0.002	,τ=1.2,ε/L=0.02)
	−4.978 ±0.002	−0.7 ±0.3	
	K=0.069, D_{KY}=2.01	K=0.073, D_{KY}=2.1	
Mackey-Glass eqs.	0.0071	0.0074±0.0007	(M=2·10^4,n=400,d=5
(a=0.2,b=0.1,c=10,	0.0027	0.0038±0.0007	,τ=24,ε/L=0.05)
T=30,δt=T/100)	0.000	−0.0015±0.0008	
	−0.0167	−0.017 ±0.003	
	−0.0245	−0.042 ±0.01	
	K=0.098, D_{KY}=3.54	K=0.112, D_{ky}=3.58	
Rayleigh-Benard experiment		0.0103±0.0009	(M=4·10^4,n=20,d=3
(Γ=2.4,P_r∿5.7,R=40.47R_c)		0.001 ±0.001	,τ=44.8,ε/L=0.04)
		−0.017 ±0.001	

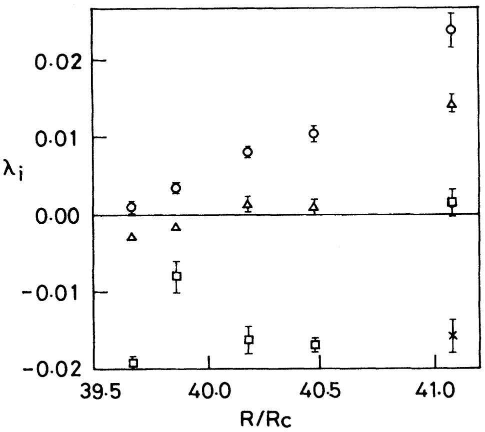

FIG. 2. The Lyapunov exponents for experimental Rayleigh-Bénard data as a function of Rayleigh number. The embedding dimension is $d=3$ for all data except for the data at the highest Rayleigh number (where $d=4$).

the attractor and M be the total number of orbital points. The number of points contained within the ball of radius ϵ among the total data points M is approximately $N \sim (\epsilon/L)^D M$. The minimum number N required to solve Eq. (9) is equal to the dimension d. Then the amount of required data M is given by $M > d(L/\epsilon)^D$. By the numerical test of our algorithm, the permissible limit on the magnitude of ϵ/L is typically 3–5%, Therefore, in the case of $d=3$, $D=2$, e.g., the above estimate gives $M > 1200$–4000, and for $d=4$ and $D=3$, e.g., $M > (3\text{–}14)\times 10^4$. The estimated amounts of data to measure or store are readily obtainable in typical experiments.[21]

We have also applied the method to experimental data of the Rayleigh-Bénard system. Our experimental setup has been described elsewhere.[21] The fluid was water at 30 °C where its Prandtl number is 5.5. We observed intermittent chaos in a $2.4\times 1.5\times 1.0\text{-cm}^3$ layer. Typical values of Lyapunov exponents obtained from time series are presented in Table I. As shown in Fig. 2, in the vicinity of the onset of chaos we obtained the Lyapunov spectrum of one positive, one zero, and one negative exponent. We denote this type of spectrum as +,0,−. With increase of the Rayleigh number the spectrum was changed into +,+,0,−. This indicates the appearance of hyperchaos. In the case of intermittent chaos, the strange attractor is not always a homogeneous fractal, so that commonly used methods to obtain the fractal dimension give no significant results. We showed that even in such cases the present method gives reliable estimates of the Lyapunov exponents. Detailed results will be published elsewhere.[20] Finally, we show in Table I the estimations of Kolmogorov entropy K by a sum of positive exponents from the spectrum,[7] and the of Kaplan-York fractal dimension D_{KY} by the use of the Kaplan-York formula,[8] respectively. The results are in agreement with the known values.

To conclude, by using the new method we could obtain good estimates of the Lyapunov spectrum from the observed time series in a very systematic way. Because of the ability of the method to measure several Lyapunov exponents, positive, zero, and even negative ones, other important characteristic invariants such as fractal dimension of attractors of Kolmogorov entropy are obtainable with great ease. It is hoped that the new method has wide applicability to systems whose dynamical equations are not available.

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