Commutator of the Quark Mass Matrices in the Standard Electroweak Model and a Measure of Maximal *CP* Nonconservation

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The structure of the quark mass matrices in the standard electroweak model is investigated. The commutator of the quark mass matrices is found to provide a convention-independent measure of *CP* nonconservation. The question of maximal *CP* nonconservation is discussed. The present experimental data indicate that nowhere is *CP* nonconservation maximal.

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One of the outstanding problems in physics is that of "explaining" the quark masses and mixings. The standard electroweak model, in spite of its dazzling empirical successes, provides only a partial solution to this problem by predicting that the quark mixing matrix is unitary. However, the values of the mixing parameters or the quark masses cannot be predicted. For example, if there are three families the mass matrices m and m' (referring respectively to charge $\frac{2}{3}$ and $-\frac{1}{3}$ quarks) are arbitrary three by three matrices. A *gedanken* diagonalization of these matrices is supposed to yield the quark masses, i.e.,

$$\begin{aligned} U_L m U_R^\dagger &= \mathrm{diag}(m_u, m_c, m_t), \\ U_L' m' U_R'^\dagger &= \mathrm{diag}(m_d, m_s, m_b), \end{aligned} \tag{1}$$

where U_x and U_x', $x = L,R$, are unitary matrices and m_j denotes the mass of the quark j. Furthermore, the unitary matrices in (1) are not measurable except for the product $U_L U_L'^\dagger$ which is the usual quark mixing matrix,[1] predicted to be unitary.

The traditional approach to the quark mass problem is to go beyond the standard model and assume specific forms for m and m' which, on subsequent diagonalization, yield the eigenvalues and the mixing parameters. Many models, based on this approach, are available in the literature and a few examples are listed in Ref. 2.

Since the quark mass matrices are of fundamental importance one would like to gain an insight into their structure without making assumptions. A first step in this direction could be to try and determine the mass matrices from experiment. Evidently the latter approach can at most be partially successful because the knowledge of the eigenvalues and the mixing parameters is not sufficient, for this purpose. Nonetheless, it is possible to make some progress[3] by noting that, in the standard model, it is always possible to go to a Hermitian basis for the mass matrices and, without loss of generality, to assume that $m = m^\dagger$ and $m' = m'^\dagger$. Then the subscripts on the unitary matrices in Eq. (1) may be dropped. Furthermore, it is convenient to normalize the mass matrices, by defining

$$M_{ij} = m_{ij}/m_t, \quad M_{ij}' = m_{ij}'/m_b, \tag{2}$$

so that the largest eigenvalue, for both mass matrices, is 1. Thus the mass matrices are related to the measurable quantities through the following relations:

$$\begin{aligned} UMU^\dagger &= D = \mathrm{diag}\left(\frac{m_\mu}{m_t}, \frac{m_c}{m_t}, 1\right), \\ U'M'U'^\dagger &= D' = \mathrm{diag}\left(\frac{m_d}{m_b}, \frac{m_s}{m_b}, 1\right), \end{aligned} \tag{3}$$

$$V = UU'^\dagger = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}. \tag{4}$$

Even in this Hermitian basis it is not possible to determine M and M'. However one may relate them to each other,[3] by eliminating, for example, the matrix U' whereby both M and M' may be expressed as functions of U and the measurable quantities. I shall return to this point later on.

A lesson learned from quantum mechanics is that the observables are represented by Hermitian operators and that their commutators give a measure of their compatibility, i.e., whether the observables can be measured simultaneously or not. The Hermitian mass matrices, in Eq. (3), are only "partial observables." Only their eigenvalues and their "relative orientation," i.e., the quantity V, are measurable. Nevertheless, their commutator is a measure of whether they can be diagonalized simultaneously or not. We shall see later on that this commutator seems to play an important role.

The commutator of the mass matrices is given by

$$[M, M'] = iC, \tag{5}$$

where C is a traceless Hermitian matrix. From Eqs.

(2)–(4) follows that

$$C = -iU^\dagger [D, VD'V^\dagger] U. \tag{6}$$

Hence, the eigenvalues of C are calculable in terms of the measurable quantities. I find that the determinant of C is given by

$$\det C = -2FF'J. \tag{7}$$

Here

$$\begin{aligned} F &= (m_t - m_c)(m_t - m_u)(m_c - m_u)/m_t^3, \\ F' &= (m_b - m_s)(m_b - m_d)(m_s - m_d)/m_b^3, \end{aligned} \tag{8}$$

and

$$J = \mathrm{Im}(V_{11} V_{22} V_{12}^* V_{21}^*). \tag{9}$$

The V_{ij} are the elements of the matrix V in Eq. (4); $V_{11} = V_{ud}$, etc.

A remarkable feature of the determinant, in Eq. (7), is that it vanishes if and only if there is no CP nonconservation. For example, if m_t were equal to m_c the CP phase could be removed, etc. Furthermore, the quantity J, in the Kobayashi-Maskawa parametrization, is given by

$$\begin{aligned} J &= s_1^2 s_2 s_3 c_1 c_2 c_3 \sin\delta, \\ s_i &= \sin\theta_i, \quad c_i = \cos\theta_i. \end{aligned} \tag{10}$$

Evidently J vanishes if any of the following conditions is satisfied

$$\theta_i = 0, \quad \theta_i = \pi/2, \quad \delta = 0, \quad \delta = \pi, \tag{11}$$

whereby there will be no CP nonconservation.

The product appearing in Eq. (10) is familiar from CP calculations. It is well known that all CP-nonconservation effects in the standard model are proportional to it. The quantity J is obtained as follows: (i) Cross out the third row and third column of the matrix V, (ii) put asterisks on the elements along the diagonal of the two by two matrix which remains, and (iii) multiply the four elements and take the imaginary part. Note that there is nothing special about the third row and the third column. There are nine different ways of crossing out one row and one column of the matrix V whereby the generic product

$$\mathrm{Im}(V_{ij} V_{kl} V_{kj}^* V_{il}^*) \tag{12}$$

is obtained. All of these quantities are equal, up to an overall sign, because V is unitary,

$$\mathrm{Im}(V_{11} V_{22} V_{12}^* V_{21}^*) = \mathrm{Im}(V_{22} V_{33} V_{23}^* V_{32}^*) = \ldots .$$

An essential feature of J, which makes it very important when discussing CP nonconservation, is that it is phase-convention independent as follows. One may redefine, at will, the quark mixing matrix by

$$V \rightarrow \phi(\alpha_1, \alpha_2, \alpha_3) V \phi(\beta_1, \beta_2, \beta_3), \tag{13}$$

where

$$\phi(\alpha_1, \alpha_2, \alpha_3) = \mathrm{diag}(e^{i\alpha_1}, e^{i\alpha_2}, e^{i\alpha_3})$$

without changing the physics. Under this transformation the elements V_{jk} change,

$$V_{jk} \rightarrow \exp[i(\alpha_j + \beta_k)] V_{jk},$$

but the product (12) remains invariant. Hence J is invariant.

The vanishing of the determinant of the commutator of the mass matrices, when there is no CP nonconservation, is a special case of a theorem which I shall discuss now.

Considering the standard model with n quark families, we may write the analog of Eqs. (2)–(4) for n families, where all the matrices are n by n.

Theorem.—For the case of n families, if the mixing matrix V is real (i.e., there is no CP nonconservation) then if r is an eigenvalue of C so also is $-r$.

Proof: C is given by

$$C = U^\dagger \hat{C} U, \quad \hat{C} = -i[D, VD'V^\dagger]. \tag{14}$$

Evidently the Hermitian matrices C and $\hat{C}$ have common eigenvalues. If V is real the matrix $\hat{C}$ is purely imaginary and therefore it is antisymmetric. Then the eigenvalue equation, $\det(\hat{C} - r\mathbf{1}) = 0$, yields immediately that $-r$ is also an eigenvalue. This completes the proof.

From the theorem it follows that if the number of families is even, $n = 2k$, and V is real, then the eigenvalues of C appear in pairs r_j, $-r_j$, $j = 1$–k, and

$$\det C = (-1)^k \prod_{j=1}^{k} (r_j)^2.$$

If, however, n is odd, at least one of the eigenvalues must vanish and, therefore, the determinant also vanishes, i.e., the commutator becomes singular.

For $n = 3$, the eigenvalues of C, in the limit of V being real, are given by

$$r_0 = 0, \quad r_\pm = \pm[\tfrac{1}{2}\mathrm{Tr}(\hat{C}^2)]^{1/2}.$$

In terms of measurable quantities we have $r_\pm \approx A\lambda^2$. Here I have used the parametrization of V *à la* Wolfenstein,[4]

$$V = \begin{pmatrix} 1 - \frac{1}{2}\lambda^2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1 - \frac{1}{2}\lambda^2 & A\lambda^2 \\ A\lambda^3(1 - \rho - i\eta) & -A\lambda^2 & 1 \end{pmatrix} + O(\lambda^4), \tag{15}$$

where

$$\lambda = V_{us} \approx 0.22, \quad A = 1, \quad \rho^2 + \eta^2 \lesssim 0.25.$$

By "turning on" the CP nonconservation the small eigenvalue moves away from zero. I find

$$r_0 \approx 2\eta\lambda^2 \frac{m_c}{m_t} \frac{m_s}{m_b},$$

which is at most of order λ^6, if $m_t \gtrsim 40$ GeV. This equation explicitly demonstrates how the small eigenvalue approaches zero as CP nonconservation is turned off, i.e., $\eta \to 0$.

The mass matrices M and M', for three families, have a very puzzling feature which was recently discovered by Frampton and the present author.[3] Using Eqs. (3), (4), and (15) we found that M and M' are equal to order λ^2,

$$M = M' + O(\lambda^2). \tag{16}$$

This empirical relation, which is independent of the Higgs structure of the standard model, indicates that the mass matrices for the up- and down-type quarks are very much "aligned." To study this alignment in more detail, I put

$$M = M' + \Delta, \tag{17}$$

where

$$\Delta = U^\dagger [D - VD'V^\dagger] U. \tag{18}$$

Thus the difference $M - M'$ is calculable, from data, up to a unitary rotation. The eigenvalues of Δ, including the order λ^2, are given by

$$\delta_0 = 0, \quad \delta_\pm = \tfrac{1}{2}[d \pm (d^2 + 4A^2\lambda^4)^{1/2}], \tag{19}$$

where $d = m_c/m_t - m_s/m_b$ is a quantity which is at most of order λ^2. Equation (19) shows that Δ is nonvanishing in order λ^2. Hence M and M' are not equal if the order-λ^2 term is included. This is interesting because in the diagonal basis for the mass matrices we have, from Eq. (3), $D = D' + O(\lambda^4)$ if the quantity d were to vanish, a possibility which is not excluded by experiment.

The above analysis of the commutator of the mass matrices which resulted in the phase-convention–independent "CP-nonconservation measure," J, gives a natural framework for discussing the question of "maximal CP nonconservation." This question is a matter of considerable current interest.[5] My approach here differs from that of the previous authors. The issue is the following. The fact that parity nonconservation is maximal in charged-current interactions is seen directly in the Lagrangean, by the presence of appropriate $1-\gamma_5$ factors. Similarly, charge-conjugation invariance is maximally violated. What should we look for in the Lagrangean in order to see whether CP nonconservation is maximal or not? How does one define maximal CP nonconservation?

In discussing this issue it is appropriate to distinguish an underlying violation of a physical principle and its manifestation in specific physical processes. For instance, in the case of parity, the amplitude for a physical process is of the generic form

$$\mathcal{M} = vV + aA, \tag{20}$$

where V (A) denotes the parity-conserving (-nonconserving) amplitude and v (a) is the relevant coupling constant. Experiment measures the parity-nonconserving quantity

$$E_P = \frac{2\,\mathrm{Re}(vVa^*A^*)}{|vV|^2 + |aA|^2}, \quad -1 \le E_P \le 1, \tag{21}$$

whereas the intrinsic parity-nonconserving parameter is

$$a_P = \frac{2\,\mathrm{Re}(va^*)}{|v|^2 + |a|^2}, \quad -1 \le a_P \le 1. \tag{22}$$

Thus, even if the intrinsic parity nonconservation would be maximal, $a_P = \pm 1$, a physical process may appear to be parity conserving if either of the amplitudes V or A should vanish.

In discussing CP nonconservation the simplest analog of the above treatment of parity is to compare the transition rates for a physical reaction and its CP-conjugate reaction. Thus, we define[6]

$$E_{CP} = \frac{\Gamma - \bar{\Gamma}}{\Gamma + \bar{\Gamma}}, \quad -1 \le E_{CP} \le 1. \tag{23}$$

In order to get an effect it is well known that at least two weak amplitudes with different phases are required. Furthermore, one must go beyond the tree approximation and include either final-state interactions or finite widths, etc., which contribute with different phases to the two amplitudes. In addition at least four different quarks must be involved. Hence the simplest fundamental case that one can consider is just a transition involving four different quarks: two up-type quarks denoted by i and k and two down-type quarks denoted by j and l. Then the transition amplitude is given by

$$\mathcal{M} = (V_{ij}V_{kl})A_1 e^{i\phi_1} + (V_{kj}V_{il})A_2 e^{i\phi_2}, \tag{24}$$

where the V's are the elements of the mixing matrix; $A_{1,2}$ denote the (real) amplitudes and $\phi_{1,2}$ are the phases due to higher-order interactions. Similarly, the amplitude for the CP-conjugate process is given by

$$\bar{\mathcal{M}} = (V_{ij}V_{kl})^* A_1 e^{i\phi_1} + (V_{kj}V_{il})A_2 e^{i\phi_2}. \tag{25}$$

Equations (24) and (25) yield

$$E_{CP} = 2\,\mathrm{Im}(\alpha\beta^*)\sin(\Delta\phi)A_1A_2\{|\alpha|^2A_1^2 + |\beta|^2A_2^2 + 2\,\mathrm{Re}(\alpha\beta^*)\cos(\Delta\phi)A_1A_2\}^{-1}, \tag{26}$$

where

$$\alpha = V_{ij} V_{kl}, \quad \beta = V_{kj} V_{il}, \quad \Delta\phi = \phi_2 - \phi_1. \tag{27}$$

Comparison with the case of parity, Eqs. (21) and (22), indicates that the appropriate definition of the intrinsic CP-nonconservation parameter is

$$a_{CP} = \frac{2\,\mathrm{Im}(\alpha\beta^*)}{|\alpha|^2 + |\beta|^2}, \quad -1 \leq a_{CP} \leq 1. \tag{28}$$

In the standard model with three families there are nine fundamental four-quark transitions (obtained from crossing out one row and one column of the matrix V). Hence there are nine different CP-nonconservation parameters of the kind given in Eq. (28). The numerators are all, up to a sign, equal to the quantity $2J$, Eq. (9).

From Eq. (28) it follows that the maximal CP nonconservation occurs iff $\alpha = V_{ij} V_{kl}$ and $\beta = V_{kj} V_{il}$ are equal in magnitude and out of phase by $\pm\pi/2$. The question is then whether this can happen in the standard model. Evidently, it is impossible to have maximal CP nonconservation in all the nine fundamental transitions. For example, if $V_{11} V_{22} = \pm i V_{12} V_{21}$ and $V_{11} V_{32} = \pm i V_{31} V_{12}$ then $V_{31} V_{22}$ and $V_{32} V_{21}$ must be relatively real. Thus for CP the situation is quite different from the case of parity where all the nine transitions $W^+ \to f\bar{f}'$ (where f and f' denote, respectively, an up-type and a down-type quark) yield maximal parity nonconservation. Another essential point is that the maximal CP nonconservation is not just a question of a relative phase assuming the value $\pm\pi/2$; the magnitudes of the coupling constants are also essential. Note also that the maximal CP nonconservation does not correspond to $\sin\delta = 1$.

A quick inspection of the parametrization (15) indicates that the CP nonconservation is large in two cases: (1) Transitions corresponding to crossing out the third row and the first column of V. These transitions which involve the quarks u, c, b, and s are characterized by the intrinsic CP-nonconservation parameter

$$a_{CP} = \frac{2\,\mathrm{Im}(V_{12} V_{23} V_{13}^* V_{22}^*)}{|V_{12} V_{23}|^2 + |V_{13} V_{22}|^2} \approx \frac{-2\eta}{1 + \rho^2 + \eta^2}. \tag{29}$$

This asymmetry could be as large as 0.8 in magnitude if $\rho \approx 0$ and $\eta^2 \approx 0.25$. (2) Transitions involving c, t, d, and s, where

$$a_{CP} = \frac{2\,\mathrm{Im}(V_{21} V_{32} V_{22}^* V_{31}^*)}{|V_{21} V_{32}|^2 + |V_{22} V_{31}|^2} \approx \frac{-2\eta}{1 + (1-\rho)^2 + \eta^2}. \tag{30}$$

An example of this kind of effect is the observed CP nonconservation in the K^0-$\bar{K}^0$ system, where the d,s are the external quarks and c,t those internal quarks which are expected to give the bulk of the contribution.

From the above analysis I conclude that if the quark mixing matrix V is the origin of CP nonconservation and the experimental data on the b lifetime are correct then nowhere is CP nonconservation maximal.

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