PHYSICAL REVIEW LETTERS

VOLUME 55 | 2 SEPTEMBER 1985 | NUMBER 10

Continuum Dissolution and the Relativistic Many-Body Problem: A Solvable Model

J. Sucher
Department of Physics and Astronomy, University of Maryland, College Park, Maryland 20742
(Received 14 June 1985)

Any attempt to describe bound states of three or more Dirac particles, or of two or more such particles in the presence of an external potential, must take into account the problem of continuum dissolution: A Hamiltonian which involves only the sum of free Dirac Hamiltonians for the particles plus local interactions will not have normalizable eigenfunctions because of the mixing of positive- and negative-energy states. A simple model is constructed to illustrate the validity of this "folk theorem," first noted by Brown and Ravenhall in 1951.

PACS numbers: 03.65.Ge, 11.10.Qr, 11.10.St, 31.30.Jv

The Dirac-Coulomb Hamiltonian for N electrons moving in an external potential $V_{\text{ext}}(\mathbf{r})$ is defined by

$$H_{\text{DC}} = \sum\nolimits_i H_{\text{D}}(i) + V_{e\text{-}e}, \tag{1a}$$

where

$$H_{\text{D}}(i) = \boldsymbol{\alpha}_i \cdot \mathbf{p}_i + \beta_i m + V_{\text{ext}}(\mathbf{r}_i) \tag{1b}$$

is a one-body Dirac Hamiltonian and

$$V_{e\text{-}e} = \sum_{i<j} \alpha / r_{ij} \tag{1c}$$

is the electron-electron Coulomb interaction. H_{DC} has been much used in the past as the starting point for the treatment of relativistic effects in atoms. However, H_{DC} is afflicted with the problem of "continuum dissolution": It has no proper (i.e., normalizable) eigenfunctions because $V_{e\text{-}e}$ has nonzero matrix elements between positive- and negative-energy eigenfunctions of the $H_{\text{D}}(i)$.[1] There has been some reluctance to accept this "folk theorem," which until a few years ago was not widely known. In view of the importance of the question it may be useful to try to remove any doubts concerning this matter, which is of relevance not only for atomic physics[2] but also for nuclear and particle physics.[3] The purpose of this note is to present a simple model, which has all the essential mathematical features of H_{DC}, but which can be used to demonstrate the point at issue in an explicit manner.

With spin ignored, consider first a one-body momentum-space Hamiltonian $H_a(1)$, acting in a space of two-component wave functions $\psi(1)$, given by

$$H_a(1) = \beta_1 E_a(\mathbf{p}_1) + \beta_1^{(+)} U_a, \tag{2a}$$

where $E_a(\mathbf{p}_1) = (m_a^2 + \mathbf{p}_1^2)^{1/2}$ and

$$\beta_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \beta_1^{(+)} = \tfrac{1}{2}(1+\beta_1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

The operator U_a is a separable potential of the form

$$U_a = g_a |v_a\rangle \langle v_a| \tag{2b}$$

or, equivalently,

$$\langle \mathbf{p}_1 | U_a | \mathbf{p}_1' \rangle = g_a v_a(\mathbf{p}_1) v_a^*(\mathbf{p}_1'),$$

with $v_a(\mathbf{p})$ a bounded and rapidly decreasing function of $\mathbf{p}^2$. A study of the eigenvalue problem

$$H_a(1)\psi(1) = E\psi(1) \tag{2c}$$

shows that for $E \geqslant m_a$, $H_a(1)$ has improper eigenfunctions of the form $\phi_{\mathbf{k}}(\mathbf{p}_1)\chi_+(1)$, where the spatial factor is a scattering-state wave function with asymptotic momentum $\mathbf{k}$, and for $E \leqslant -m_a$ the improper eigenfunctions are pure plane waves of the form $\delta(\mathbf{p}_1 - \mathbf{k})\chi_-(1)$; here

$$\chi_+(1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_-(1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Provided that $g_a < 0$, there is also a proper (normalizable) eigenfunction of the form $\phi_a(\mathbf{p}_1)\chi_+(1)$ with eigenvalue E_a', in the interval $0 < E < m_a$. Using standard techniques one can show that E_a' is the unique solution of the equation

$$-(1/g_a) = \langle v_a|(E_a - E)^{-1}|v_a\rangle = \int d\mathbf{p}|v_a(\mathbf{p})|^2/[E_a(\mathbf{p}) - E]^{-1}.$$

Thus, apart from the presence of only one bound state, the spectrum of $H_a(1)$ has the same character as that of $H_D(1)$: continuous for $E > m_a$ and $E < -m_a$, discrete in the interval $0 < E < m_a$.

The spectrum of the sum H_0 of two such operators,

$$H_0 = H_a(1) + H_b(2), \tag{3}$$

is similar in character to that of the sum $H_D(1) + H_D(2)$: It is the whole real line, with a single bound state

$$\psi_0(1,2) = \phi_a(\mathbf{p}_1)\phi_b(\mathbf{p}_2)\chi_{++}, \tag{4}$$

of energy $E_0 = E_a' + E_b'$, embedded in this continuum. Here I have introduced the abbreviation

$$\chi_{rs} = \chi_r(1)\chi_s(2).$$

The states degenerate in energy with ψ_0 include those of the form $\delta(\mathbf{p}_1 - \mathbf{k}_1)\phi_{\mathbf{k}_2}(\mathbf{p}_2)\chi_{-+}$, with $\mathbf{k}_1$ and $\mathbf{k}_2$ satisfying the condition $E_0 + E_a(\mathbf{k}_1) - E_b(\mathbf{k}_2) = 0$.

Consider next an interaction V between "1" and "2" which mixes positive- and negative-energy continuum eigenfunctions of H_0. A simple choice is

$$V = \rho_1\beta_2^{(+)}U, \quad \rho_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \beta_2^{(+)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{5}$$

with the operator U acting only on spatial coordinates; it need not be further specified at this stage. It is straightforward to show that with

$$H = H_0 + V, \tag{6a}$$

the eigenvalue equation

$$H\psi(1,2) = E\psi(1,2) \tag{6b}$$

does not have normalizable solutions.

To see this, write ψ in the form

$$\psi = \phi_{++}\chi_{++} + \phi_{-+}\chi_{-+} + \phi_{+-}\chi_{+-} + \phi_{--}\chi_{--}, \tag{7}$$

substitute into (6b), and take the scalar product with χ_{++} and with χ_{-+}. This yields a pair of coupled equations which may be written in the form

$$(E - E_a - U_a - E_b - U_b)\phi_{++} = U\phi_{-+}, \tag{8a}$$

$$(E + E_a - E_b - U_b)\phi_{-+} = U\phi_{++}. \tag{8b}$$

The functions ϕ_{+-} and ϕ_{--} are decoupled and may be set equal to zero. Since

$$U_b\phi_{-+} = g_b v_b(\mathbf{p}_2)f(\mathbf{p}_1),$$

where

$$f(\mathbf{p}_1) = \langle v_b|\phi_{-+}\rangle = \int d\mathbf{p}_2 v_b^*(\mathbf{p}_2)\phi_{-+}(\mathbf{p}_1,\mathbf{p}_2), \tag{9}$$

Eq. (8b) yields the relation

$$\phi_{-+}(\mathbf{p}_1,\mathbf{p}_2) = N/D, \tag{10a}$$

with

$$N = U\phi_{++}(\mathbf{p}_1,\mathbf{p}_2) + g_b f(\mathbf{p}_1)v_b(\mathbf{p}_2), \tag{10b}$$

for all values of $\mathbf{p}_1$ and $\mathbf{p}_2$ such that

$$D = D(\mathbf{p}_1,\mathbf{p}_2;E) = E + E_a(\mathbf{p}_1) - E_b(\mathbf{p}_2) \neq 0. \tag{10c}$$

On substituting the form (10a) into (9) and solving for $f(\mathbf{p}_1)$ we get

$$f(\mathbf{p}_1) = [1 - g_b\langle v_b|D^{-1}|v_b\rangle]^{-1} \times \langle v_b|D^{-1}|U_{\phi_{++}}\rangle. \tag{10d}$$

The integration over the singularity at $\mathbf{p}_2^2$ such that $D = 0$ is taken to be a principal value, but another choice such as an $i\epsilon$ prescription will not change the conclusion. From Eqs. (10a)–(10d) we see that for $D \neq 0$ the function $\phi_{-+}(p_1,p_2)$ is completely determined by ϕ_{++} and the choice of integration path.

Now examine the behavior of ϕ_{-+} in the neighborhood of the region $D = 0$. One may convince oneself that the numerator function N does not vanish identically on the surface $D = 0$, for any choice of E.[4] It follows that the integral

$$\langle\phi_{-+}|\phi_{-+}\rangle_\epsilon = \iint_{|D|>\epsilon} d\mathbf{p}_1 d\mathbf{p}_2 |N/D|^2 \tag{11}$$

diverges like $1/\epsilon$ as $\epsilon \to 0$. Hence, regardless of how the function ϕ_{-+} is defined in the $D = 0$ manifold, the norm of ϕ_{-+} will be infinite. Since the spinors χ_{rs} are orthogonal, we have

$$\langle\psi|\psi\rangle = \langle\phi_{++}|\phi_{++}\rangle + \langle\phi_{-+}|\phi_{-+}\rangle, \tag{12}$$

so that the norm of ψ itself is infinite for any choice of E, even if the norm of ϕ_{++} is finite.

This completes the proof. The equation for ϕ_{++} is still complicated even for simple choices of U, so that in this sense the model is not fully solvable, but as the above analysis shows there is no need to have an analytic form for ϕ_{++} to illustrate the point in question. Finally, if there are only two particles and no external field the problem disappears because of momentum conservation. For the example at hand, the dangerous denominator then has the form, in the c.m. system, $D = E + E_a(\mathbf{p}) - E_b(-\mathbf{p})$, which reduces to E in the equal-mass case, but which has no zeros even in the general case provided that the binding energy does not exceed $2m_b$.

This work was supported in part by the National Science Foundation.

[1]G. E. Brown and D. G. Ravenhall, Proc. Roy. Soc. London, Ser. A **208**, 552 (1951).

[2]For a review see, e.g., J. Sucher, Int. J. Quantum Chem. **24**, 3 (1984), and invited talk, Atomic Theory Workshop on Relativistic and QED Effects in Heavy Atoms, Gaithersburg, Maryland, 23–24 May 1985 (to be published).

[3]For example, any attempt to study relativistic effects in a three-body bound state such as the triton with a Hamiltonian which is the sum of three free Dirac Hamiltonians plus a local interaction between the nucleons is bound to founder, as would an analogous attempt to study such effects for the proton, considered as a bound state of three quarks. Configuration-space Hamiltonians obtained from quantum field theory invariably involve positive-energy projection operators surrounding the interparticle potentials, so that continuum dissolution is avoided.

[4]One way to do this is to consider the case $U = gP_aP_b$, where P_a and P_b are projections onto the bound states ϕ_a and ϕ_b, respectively, and to study N for small values of g, when ϕ_{++} may be approximated by $\phi_a\phi_b$.