Induced Chern-Simons Terms at High Temperatures and Finite Densities

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The *CP*-nonconserving portion of the gauge-field effective action for an even number of left-handed SU(2) fermion doublets, with nonzero chemical potentials μ^i, is calculated to leading order in the inverse temperature, T^{-1}. It is shown to be $i\sum_i(\mu^i/T)\,W[A]$, where $W[A]$ is the Chern-Simons topological mass term. This result is also shown to be a consequence of the $\mathrm{U}(1)_L$ anomaly at zero temperature, but with $\mu\neq 0$. Because of the i in front of the Chern-Simons term, this produces magnetic screening only if μ is complex.

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At high temperatures, a relativistic quantum field theory becomes effectively three dimensional.[1-4] In constructing the effective three-dimensional theory one first writes the thermal partition function as a Euclidean path integral in which the imaginary time variable, τ, runs from 0 to β. At high temperatures, fermionic and nonstatic bosonic modes—those which depend upon τ—acquire masses proportional to the temperature, T, and decouple, leaving behind effective interactions for the remaining static fields. Ordinarily, the only effect of the fermionic and nonstatic bosonic modes is to renormalize the effective theory[5] without introducing any qualitatively new interactions.

In this Letter, we discover an exception to this rule: We find that certain *P*- and *CP*-nonconserving effects in four-dimensional gauge theories at high temperatures can induce a topological term—proportional to the Chern-Simons secondary characteristic class—in the action for the effective three-dimensional gauge fields. The Chern-Simons term, $W[A]$, has been studied extensively in the context of three-dimensional Yang-Mills theory,[6,7] and it has been suggested[7,8] that it would play a role in the effective three-dimensional Yang-Mills theory at high temperatures. It has also been shown that $W[A]$ can be induced by fermions in three dimensions.[9] We show here for the first time, however, that the Chern-Simons term can be induced in a physically realistic system at high temperatures. In addition to being *P* and *CP* nonconserving, $W[A]$ is invariant under infinitesimal gauge transformations, but changes by an integer, n, under a homotopically nontrivial gauge transformation with winding number n.[7]

Three-dimensional Yang-Mills theory with a Chern-Simons term has another important and well-known property[7]: If the coefficient in front of the Chern-Simons term is α then the gauge fields become massive, with the square of the mass proportional to α^2. Since we are interested here in the effective three-dimensional theory for the spatial (magnetic) components of the gauge field at high temperatures, the Chern-Simons term produces magnetic screening (magnetic mass) for $\alpha^2>0$ and antiscreening for $\alpha^2<0$. Although the fermionic and nonstatic bosonic modes are known to produce an electric mass, or inverse screening length (Debye screening) for the time component of the static gauge field, A_0, no such magnetic mass has been seen in perturbation theory. The existence of a magnetic mass is considered highly desirable as it would (1) serve as an infrared cutoff which would help eliminate the severe infrared infinities encountered in a perturbative expansion of the effective three-dimensional Yang-Mills theory[10] and (2) force magnetic flux to be confined (as long as flux is conserved, $\nabla\cdot\mathbf{B}=0$, as it is in the effective theory discussed here) which it is believed would enhance monopole-antimonopole annihilation at high temperatures, thus offering an alternative solution to the monopole problem in the early universe.[11]

While we believe that many different types of *P* and *CP* nonconservation may generate a Chern-Simons term, we restrict ourselves to *P* and *CP* nonconservation induced by coupling of the gauge fields to chiral (left- or right-handed) fermions with finite chemical potentials—at finite density. Chiral fermions do not conserve *P* and *C* but conserve *CP*. To introduce *CP* nonconservation, we include a chemical potential, μ, which is the Lagrange multiplier for the conserved particle-number operator $Q_L=\int d^3x\,\bar{\psi}_L\gamma^0\psi_L$ (we ignore nonconservation due to instantons, because these effects are negligible in all realistic theories[12]). Since Q_L is odd under *CP*, and since μ, which is just a number, does not transform under *CP*, our theory effectively violates both *P* and *CP* symmetry. For μ

real, μQ_L is also odd under *CTP*. It must be stressed that we have introduced the chemical potential in the ordinary way. The *effective* nonconservation of *CP* and *CTP* occurs in the thermal partition function and does not imply that *CP* or *CTP* symmetry is violated in the underlying particle interactions.

Our investigations reveal that when *P*, *CP*, and *CTP* symmetries are all effectively violated—as they are when a real chemical potential is included—the coefficient, α, of the Chern-Simons term is imaginary, giving $\alpha^2 < 0$: antiscreening. If we effectively violate only *P* and *CP* symmetry, however, α is real and we get magnetic screening. Unfortunately, in the particular system under study here, we can restore *CTP* invariance and thereby obtain magnetic screening only by continuing the parameter α—which in our case is proportional to *i* times the chemical potential—to an unphysical value— an imaginary chemical potential.

For definiteness, we consider a four-dimensional theory with an even number of left-handed fermions in the fundamental representation of $SU(2)_L$, interacting with SU(2) gauge fields—as in the standard model of the weak interactions. Such a theory is free of both triangle anomalies,[13] and of the nonperturbative Witten SU(2) anomaly.[14] The thermodynamic partition function may be written as a Euclidean functional integral

$$Z=\int dA\, d\bar{\psi}\, d\psi \exp\left\{-\int_0^\beta d\tau \int d^3x\left[\tfrac{1}{2}\mathrm{tr}F^2+\textstyle\sum_i \bar{\psi}^i_L(i\partial\!\!\!/ - gA\!\!\!/ + i\mu_i\gamma^0)\psi_L\right]\right\}, \tag{1}$$

$$\beta=1/T,\quad A^\mu=\tfrac{1}{2}A^\mu_a\sigma_a,\quad F^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu-ig[A^\mu,A^\nu],\quad \psi_L=\tfrac{1}{2}(1+\gamma_5)\psi,$$

over gauge fields which are periodic in Euclidean time, $A(0,\mathbf{x})=A(\beta,\mathbf{x})$, and fermion fields which are antiperiodic, $\psi^i(0,\mathbf{x})=-\psi^i(\beta,\mathbf{x})$—$i$ runs from 1 to N, N even[3]; σ_a are the Pauli matrices. Integrating out the fermion fields we obtain an effective action for the gauge fields:

$$I[A]=\int_0^\beta\int d^3x\,\tfrac{1}{2}\,\mathrm{tr}F^2+\textstyle\sum_i \ln\det(i\partial\!\!\!/ - gA\!\!\!/ + i\mu_i\gamma^0),$$

where

$$\ln\det(i\partial\!\!\!/ - gA\!\!\!/ + i\mu_i\gamma^0)=I^i_{CP}[A]+I^i[A],$$

with I^i_{CP} odd under *P* and *CP*, and I^i even. For simplicity of notation, the superscript i will be suppressed until the end of the calculation.

We now use perturbation theory to calculate I_{CP} at high temperatures to leading order in $1/T$. In perturbation theory

$$I_{CP}[A]=\tfrac{1}{2}\int_0^\beta d\tau\int d^3x\,A^\mu_a(\mathbf{x},\tau)\int_r e^{-ir\cdot x}G^{\mu\nu}_{ab}(r)\hat{A}^\nu_b(r)$$
$$+\tfrac{1}{3}\int_0^\beta\int d^3x\,A^\mu_a(\mathbf{x},\tau)\int_{r,s}e^{-i(r+s)\cdot x}G^{\mu\nu\alpha}_{abc}(r,s)\hat{A}^\nu_b(r)\hat{A}^\alpha_c(s)+O(1/T), \tag{2}$$

$$G^{\mu\nu}_{ab}(r)=(-2g^2)\epsilon^{\mu\nu\alpha\rho}r^\alpha\,\mathrm{tr}\left[\frac{\sigma_a}{2}\,\frac{\sigma_b}{2}\right]\int_p p^\rho S_\beta(p;\mu)S_\beta(p;\mu)+O(1/T), \tag{3a}$$

$$G^{\mu\nu\alpha}_{abc}(r,s)=(-ig^3)\epsilon^{\mu\nu\alpha\rho}\,\mathrm{tr}\left[\frac{\sigma_a}{2}\,\frac{\sigma_b}{2}\,\frac{\sigma_c}{2}\right]\int_p p^\rho S_\beta(p;\mu)S_\beta(p;\mu)S_\beta(p;\mu)+O(1/T), \tag{3b}$$

where $G^{\mu\nu}_{ab}$ and $G^{\mu\nu\alpha}_{abc}$ are the *CP*-odd portions of the two-point and three-point Green's functions, $ip\!\!\!/S_\beta(p;\mu)$ is the propagator at temperature T and chemical potential μ, and $\hat{A}(r)$ is the Fourier transform of $A(x)$. The integral $\int_p$ in (3) is actually a combined sum and integral $[1/(-i\beta)]\sum_n\int d^3p/(2\pi)^3$ in Euclidean space. It has been proven that all of the infinities at finite T,μ can be eliminated by renormalization of the theory at $T=0$, $\mu=0$. Therefore, to extract the finite portions of our graphs, we must separate our calculation into $T=0$, $\mu=0$ and $T\neq0$, $\mu\neq0$ parts. To do so, it is easier to continue τ to the real-time interval $-\infty<x_0<\infty$, in which case the sum $[1/(-i\beta)]\sum_n$ becomes an integral.[2] The propagator $ip\!\!\!/S_\beta(p;\mu)$ will denote either the real or the imaginary time propagator; we will make clear which we are using.

Observe that $G^{\mu\nu}(r)$ at zero temperature (and $\mu\neq0$) must vanish because the only *P*-odd Lorentz-invariant function of r with the correct dimensions is $\epsilon^{\mu\nu\alpha\beta}r^\alpha r^\beta=0$. At finite temperatures, however, Lorentz invariance is broken by the heat-bath vector u^μ which we have chosen here equal to (1,0,0,0). This opens the possibility that $G^{\mu\nu}$ is proportional to $\epsilon^{\mu\nu\alpha\beta}r^\alpha u^\beta$. Still, with zero chemical potential, the propagator (in real time),[2]

$$ip\!\!\!/S_\beta(p;0)=\frac{i}{p\!\!\!/+i\epsilon}+\frac{2\pi\delta(p^2)}{e^{\beta|p\cdot z|}+1},$$

is a function of the absolute value of u and therefore $G^{\mu\nu}$ cannot depend on the sign of u, i.e., cannot be proportional to $\epsilon^{\mu\nu\alpha\beta} r^\alpha u^\beta$. Only when we include a chemical potential can $G^{\mu\nu}$ be nonvanishing, since the real-time propagator with $\mu \neq 0$ becomes[15,16]

$$ip S_\beta(p;\mu) = \frac{i}{p + i\epsilon} + 2\pi\delta(p^2)\left\{\frac{\theta(p\cdot u)}{e^{\beta(|p\cdot u| - \mu)} + 1} + \frac{\theta(-p\cdot u)}{e^{\beta(|p\cdot u| + \mu)} + 1}\right\}. \tag{4}$$

In calculating (3) using real-time propagators, one cannot simply multiply the propagators (4) because one encounters undefined double and triple delta functions. This problem can be circumvented by using a 2×2 matrix formalism,[16] or by inserting a mass m and taking derivatives of $S_\beta(p;\mu;m)$ with respect to m as suggested in Ref. 2. Using either method, and performing the integrals in (3), we find (for the real-time quantities)

$$G_{ab}^{\mu\nu}(r) = \left(ig^2\frac{\mu}{4\pi^2}\right)\epsilon^{\mu\nu\alpha 0} r^\alpha \operatorname{tr}\left[\frac{\sigma_a}{2}\frac{\sigma_b}{2}\right] + O(1/T), \tag{5a}$$

$$G_{abc}^{\mu\nu\alpha} = \left(-ig^3\frac{\mu}{8\pi^2}\right)\epsilon^{\mu\nu\alpha 0} \operatorname{tr}\left[\frac{\sigma_a}{2}\frac{\sigma_b}{2}\frac{\sigma_c}{2}\right] + O(1/T). \tag{5b}$$

Before proceeding to insert (5) into (3) and then (2), we can further simplify $I_{CP}[A]$ by writing it as a function of static gauge fields only: gauge fields which do not depend on Euclidean time. This approximation is justified at high temperatures, since in a mode expansion,

$$A^\mu(\mathbf{x},\tau) = \sum_n A_n^\mu(\mathbf{x})\exp[i(2\pi n/\beta)\tau],$$

one finds that all the modes other than the static mode, $n=0$, acquire masses of order T. Except for coupling-constant renormalization, these can be ignored in the effective three-dimensional theory which is valid for length scales $l >> 1/T$. [We shall use $g = g(T,\mu)$ to denote the renormalized coupling constant at temperature T and chemical potential μ.]

Letting $A^\mu(\mathbf{x},\tau) \to A_0^\mu(\mathbf{x},0)/\sqrt{T} \equiv A'^\mu(\mathbf{x})$, and using (5) in (3) and (2), we find

$$I_{CP} = i(\mu/T)[A'] + O(1/T), \tag{6}$$

$$W[A'] = \frac{ie^2}{16\pi^2}\epsilon^{ijk}\, tr[A_i' F_{jk}' - i\tfrac{2}{3} e A_i' A_j' A_k'], \tag{7}$$

where $W[A']$ is the Chern-Simons term. In three dimensions, the coupling constant is $e = g\sqrt{T}$, and for sufficiently high density we expect μ/T to be of order one.

The Chern-Simons term in (6) may also be derived in Minkowski space at zero temperature for $\mu \neq 0$. At zero temperature, we cannot neglect fermion masses as we did at high temperatures. Instead, we include the right-handed components of the fermion fields and give the fermion doublets masses, m_i. The term $\sum_i \mu^i \bar{\psi}^i \gamma^0 \psi^i$ can be interpreted as the coupling of an external U(1) gauge field $B_i^\mu = (\mu_i, 0,0,0)$ to the U(1) current $J^\mu = \bar{\psi}^i \gamma^\mu \psi^i$. In the presence of the $SU(2)_L$ gauge fields, this U(1) current is not conserved, because of the $U(1)_L$ anomaly. As a consequence, $\langle J^\mu \rangle$ does not vanish. By expanding in powers of the momenta p over masses m_i, we may use a well-known result[17] for $\langle J^\mu \rangle$ in the presence of $SU(2)_L$ gauge fields. We find

$$I_{CP} = \int d^4x\, \tfrac{1}{2} B_i^\mu \langle J_\mu^i \rangle$$
$$= \sum_i \mu^i \int_{-\infty}^{\infty} dx_0 (-i)\, W[A] + O(p/m_i),$$

where in Minkowski space we include an integral over x_0—the field $A(x_0)$ is the four-dimensional field, with dimension T, and $W[A]$ is given by (7) with $e \to g$. This proves that the Chern-Simons term is present even at $T=0$, and its existence is a consequence of the $U(1)_L$ anomaly. However, it is only for high temperatures that we can ignore gauge fields which are τ dependent and discuss $\mu \int dx_0(-i)\, W[A] \to (\mu/T)\, W[A']$ as a mass term in the context of an effective three-dimensional theory.

As mentioned above, the Chern-Simons term (7) is not gauge invariant, but changes by n under a homotopically nontrivial gauge transformation with winding number n, n integer. Requiring the phase exponential of the action to be gauge invariant leads to a quantization condition[7]: $i(1/2\pi T)|\sum_i \mu^i| = n$. Unless the μ^i are imaginary, the quantization condition cannot be satisfied. However, we are not convinced that it must be satisfied here. In introducing the chemical potential, we have ignored particle-number–nonconserving effects due to instantons. This is a justified approximation since these effects produce decay rates which are negligibly small in all physically interesting theories.[12] It may be that ignoring the tunneling effects of four-dimensional instantons is equivalent to restricting the effective three-dimensional theory to a single homotopy class. If this is so, we should not be surprised to find that our quasiequilibrium effective theory is not invariant under large gauge transformations. We have also ignored possible terms in the effective theory which would not be seen in perturbation theory; these might restore gauge invariance.

The generation of a Chern-Simons term in the effective three-dimensional theory at high temperatures and finite densities is a physical effect (for μ real) which should have measurable consequences. It is not entirely clear to us what these consequences are, although we speculate that the antiscreening ($\alpha^2 < 0$) in the effective theory is a signal of instability. To see that we are discussing a potentially significant physical effect, however, let us calculate the magnitude of the coefficient of the Chern-Simons term, $|\alpha|$, during an intermediate stage in the evolution of the early universe. The magnitude of α, a dimensionful quantity, is a measure of the screening ($\alpha^2 > 0$, μ imaginary) or antiscreening ($\alpha^2 < 0$, μ real) at high temperatures. Comparing the result of Ref. 7 to Eq. (6), we learn that $|\alpha| = (g^2/4\pi^2)|\sum_i \mu^i|$. For $|\alpha|$ to be significant we must have a large particle-antiparticle asymmetry. One place where such an asymmetry may exist is in the present universe. Available upper bounds on neutrino-antinueutrino asymmetry and on muon number give chemical potentials which go like T times some number of order one: $\mu \sim T$. If a CP-nonconserving interaction at the grand-unification-theory scale was responsible for this asymmetry, then at some later time when the universe was at a temperature 10^9 GeV, we may expect the interactions which lead to particle-number nonconservation to slow to a point where particle number becomes approximately frozen. Then we may apply the relationship $\mu \sim T$. At 10^9 GeV, $|\alpha| \cong (g^2/4\pi^2) \times 10^9$ GeV $\cong 5 \times 10^6$ GeV which is much greater than the electroweak scale, 100 GeV, and should produce a significant effect—we have used $g^2(10^9 \text{ GeV}) \cong \frac{1}{50}$.

We have also demonstrated that if P and CP symmetries alone are violated, real magnetic screening would result—although in our case, this requires an unphysical imaginary chemical potential. One might, of course, obtain partial magnetic screening by giving μ or g^2 an imaginary part (by g^2, we do not mean $|g|^2$). It is plausible, however, that some truly physical effect, which only violates P and CP symmetries might also induce magnetic screening. Such an effect, with $\mu = 0$, cannot generate the Chern-Simons term in perturbation theory, because the propagators would not depend on the sign of the heat-bath vector, u^α. Magnetic screening for $\mu = 0$ might still be generated by nonperturbative CP-nonconserving effects such as by instantons in a θ vacuum. It might also be generated in the presence of background fields with $E \cdot B \neq 0$. Finally, we offer the wild possibility that the infrared instability of the effective three-dimensional gauge theory might itself cause a spontaneous breakdown of CP symmetry with the resulting generation of a magnetic mass.

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