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Quantum Gravity in Two Dimensions: Exact Solution of the Jackiw Model

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The two-dimensional theory of gravity proposed by Jackiw is exactly quantized in the open case. The Wheeler-DeWitt equation is solved in closed form with and without cosmological constant. The theory has no degree of freedom and the unique wave functional reflects the classical aspects of the solution.

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Lower-dimensional theories of gravity have attracted a growing interest recently.[1] This is not surprising in view of the apparently insurmountable difficulties encountered in the quantization of the gravitational field in four dimensions. In order to get insight into these difficulties, it is then natural to turn to simpler models obtained by the lowering of the number of dimensions, which still share interesting features with their four-dimensional counterpart. A similar step has proved very useful in the study of Yang-Mills theories.[2]

In two dimensions, the Riemann tensor is completely determined by the curvature scalar ${}^{(2)}R$. Hence, the natural analog of the vacuum Einstein equations with a cosmological constant Λ is[1,3–5]

$$ {}^{(2)}R - 2\Lambda = 0. \qquad (1)$$

In order to go to the quantum theory, it is desirable to derive the equation (1) from a local action principle. Since this equation is invariant under arbitrary changes of space-time coordinates, the action should preferably be also invariant. This requirement of general covariance appears necessary if one wants to mimic the four-dimensional theory of gravity.

The only variational principle which possesses this property seems to be the one proposed by Jackiw,[4] which is adopted here.[6] The action reads

$$S[g_{\alpha\beta},\phi] = \int d^2x \sqrt{-g}\,\phi({}^{(2)}R - 2\Lambda), \qquad (2)$$

where ϕ is a scalar field. Because of this extra field, the theory has superficially zero degrees of freedom, and not "minus one," which renders its quantization more interesting.

The equations for the field ϕ, obtained by variation of the action (2) with respect to the metric components, are equivalent to

$$\nabla_\alpha \nabla_\beta \phi + \Lambda g_{\alpha\beta}\phi = 0, \qquad (3)$$

where ∇_α stands for the space-time covariant derivative. Two of these equations, namely, the $g_{0\beta}$ equations, are constraints on the initial data. The remaining g_{11} equation is truly dynamical.

As a result of general covariance, the Hamiltonian derived from (2) is very similar to the Dirac-Arnowitt-Deser-Misner Hamiltonian of four-dimensional gravity.[7–9] It is a linear combination of the above-mentioned constraints (rewritten in terms of the canonical variables):

$$\mathscr{H} = -Pg^{1/2} + 2(g^{-1/2}\phi')' + 2\Lambda\phi g^{1/2} \approx 0, \qquad (4a)$$

$$\mathscr{H}_1 = -2P'g - Pg' + p\phi' \approx 0, \qquad (4b)$$

and, hence, it vanishes weakly, as indicated. In (4), g is the spatial metric g_{11}, P is its conjugate momentum, whereas p is the momentum conjugate to ϕ. As usual, spatial derivatives are denoted with a prime.

If the spatial sections are open, which is the case considered in this paper, the equations of motion (1) and (3) must be supplemented by appropriate boundary conditions at spatial infinity (this is the standard viewpoint). These asymptotic conditions are not entirely dictated by the theory itself, but cannot nevertheless be chosen at will. Indeed, they must fulfill at least two consistency requirements: (i) They should not be contradictory with the equations of motion, i.e., they should be compatible with at least one solution. (ii) They should make the Hamiltonian generator of time translations well defined.[10]

Because the choice of the proper boundary conditions involves some subtle points, it is discussed in some detail here. Let us first consider the case of a vanishing cosmological constant, and then state the results in the general case.

When Λ is zero, the general solution to the equations of motion (1) and (3) reads, in Minkowskian coordinates,

$$g_{\lambda\mu} = \eta_{\lambda\mu} = \mathrm{diag}(-,+), \tag{5a}$$

$$\phi = a_\mu X^\mu + b. \tag{5b}$$

The numbers a_μ and b are integration constants.

From (5b), one learns two things. First of all, the field equations admit not one, but many different solutions. These can be parametrized by the invariant

$$C = -a_\mu a^\mu = -\nabla_\mu \phi \nabla^\mu \phi \tag{6}$$

when $a_\mu \neq 0$ and by b when $a_\mu = 0$. (When $a_\mu \neq 0$, b can be set equal to zero without loss of generality.) Second of all, when $a_\mu \neq 0$, the scalar field ϕ superficially breaks the Lorentz invariance.[11]

I will adopt in this note boundary conditions which guarantee that the theory has exactly zero degrees of freedom, i.e., that the solution of Eqs. (1) and (3) is unique up to a coordinate transformation. This is done by freezing of C to a given value and is motivated by the fact that the "ancestor" of the theory, namely, three-dimensional gravity,[1] has no degree of freedom in the open case. Accordingly, it seems legitimate also to enforce this property after dimensional reduction.

Moreover, since the scalar field ϕ superficially breaks the global Lorentz invariance, I will not try to work with boundary conditions which contain all solutions (5b) compatible with a given C. Rather, boundary conditions are adopted which are themselves not invariant under asymptotic Lorentz rotations, in which a_μ has only one nonvanishing component.[12] These read explicitly

$$\Lambda = 0, \quad g \to 1, \quad \phi \to \alpha, \tag{7a}$$

$$C = a^2 > 0, \quad p \to 0, \quad P \to -a. \tag{7b}$$

The precise rate of approach of the canonical variables to their asymptotic form will not be needed here. Let us insist that in (7), a is a fixed constant which characterizes the model, so that one has actually many different theories, one for each value of a. In contrast, the constant α depends on the slice under consideration and, hence, is not fixed. For simplicity, it has been assumed that $C > 0$. The other cases are treated in a similar way.

One can repeat the above analysis when the cosmological constant does not vanish. The appropriate boundary conditions turn out to be

$$\Lambda = \frac{1}{R^2} > 0, \quad g \to \cosh^2\alpha, \quad \phi \to \sinh\alpha, \tag{8a}$$

$$p \to -\frac{2}{R}\sinh\alpha, \quad P \to -\frac{1}{R}, \tag{8b}$$

$$\Lambda = -\frac{1}{R^2} < 0, \quad g \to 1, \quad \phi \to \cosh\frac{x}{R}\sin\alpha, \tag{9a}$$

$$p \to 0, \quad P \to -\frac{\cos\alpha}{R}, \tag{9b}$$

where I have restricted, for simplicity, the value of the constant C, which is now given by

$$C = -\nabla_\mu\phi\nabla^\mu\phi - \Lambda\phi^2, \tag{10}$$

to be $1/R^2$. (Again, the other cases are treated in a similar fashion.)

It is easy to check that with the boundary conditions (7)–(9), the generator of time translations $\int N\mathscr{H}\,dx$, with $N \to 1$ for $\Lambda \geqslant 0$ and $N \to \cosh(x/R)$ for $\Lambda < 0$, is well defined without the need to be improved by the addition of appropriate "surface terms" at infinity.[10] Accordingly, the associated "charge" is zero, which means that no "energy" is associated with the scalar field ϕ, even when it does not vanish.

In the approach to quantization where one quantizes without fixing the gauge—and this appears to be the only interesting one here since if one fixed the gauge and only kept the true physical degrees of freedom, nothing would be left—one imposes the equations (4) as operator constraints on the physical states,[9]

$$\hat{\mathscr{H}}\,|\psi\rangle = 0, \quad \hat{\mathscr{H}}_1\,|\psi\rangle = 0. \tag{11}$$

In the metric representation, the states are functionals of $g(x), \phi(x)$, and the momenta become

$$\hat{P}(x) = \frac{\hbar}{i}\frac{\delta}{\delta g(x)}, \quad \hat{p}(x) = \frac{\hbar}{i}\frac{\delta}{\delta\phi(x)}. \tag{12}$$

The equations (11) are then called the "Wheeler-DeWitt equation."[13] It turns out that these equations can be solved exactly in two dimensions, contrary to what seems to happen in three (and more!) dimensions, where only incomplete[14] or approximate[15] results are known.

The difficulties with the equations (11) as they stand are at least twofold. First of all, there is a factor-ordering ambiguity in both $\hat{\mathscr{H}}$ and $\hat{\mathscr{H}}_1$ because the constraints involve products of noncommuting variables. Second of all, since $\hat{\mathscr{H}}$ contains the second functional derivatives $\delta^2\psi/\delta g(x)\delta\phi(x)$ evaluated at the same space point, one expects that ill-defined expressions like $\delta(0)$ will occur in (11), which will accordingly need an appropriate regularization.

Now, contrary to what happens in ordinary field theory where the divergences in the Hamiltonian are associated, for instance, with the zero-point energy of the vacuum fluctuations, there does not seem to be any good physical reason for the occurrence of $\delta(0)$ here, which appears to be a pure gauge effect. Indeed, the present model is purely kinematical and one feels accordingly that the true physical "zero-point energy," say, should be zero.

One way of regularizing of the quantum equations with this observation in mind is to replace (before going to the quantum theory) the Hamiltonian constraints by equivalent constraints which are solved for the momenta $p(x)$ and $P(x)$. In that representation of the constraints, one simply avoids $\delta(0)$ since no second-order functional derivatives $\delta^2/\delta\phi(x)\delta g(x)$ arise in the equations. There is, in addition, no factor-ordering problem, and hence one gets at the same time a prescription for the ordering of the Wheeler-DeWitt equation in its original form, which must be handled in such a way that the quantum theory based upon (11) is equivalent to the one defined below for which the quantum constraints are consistent.

Solving the Dirac-Arnowitt-Deser-Misner constraints of general relativity for the momenta conjugate to the "pure gauge" field components is an old goal of the canonical approach to quantum gravity.[16] That it can be carried through here is an interesting feature of the model, which makes its exact quantum resolution possible.

The explicit transformation of the constraints (4) proceed as follows. If one multiplies (4b) by P and uses (4a) to eliminate p, one gets the differential equation

$$-(P^2 g)' + [(\phi' g^{-1/2})^2 + \Lambda\phi^2]' \approx 0, \tag{13}$$

which straightforwardly yields

$$P \approx -g^{-1/2}[(\phi' g^{-1/2})^2 + \Lambda\phi^2 + C]^{1/2}. \tag{14}$$

In (14), C is the (given) integration constant introduced above [Eq. (10)]. To go from (13) to (14), I have used explicitly the boundary conditions (7)–(9), and I have everywhere oriented for convenience the normal to the spacelike slices so that $P \leqslant 0$. Otherwise, one would have to keep track of the sign in front of the square root in (14) as one moves from $x = -\infty$ (where $P < 0$) and crosses the zeros of P. I have thus also implicitly assumed that $a > 0$ in (7) and that $-\pi/2 \leqslant \alpha < \pi/2$ in (9).

From Eqs. (4a) and (14), one then gets the momentum p as

$$p \approx -[(\phi' g^{-1/2})^2 + \Lambda\phi^2 + C]^{-1/2}[2(g^{-1/2}\phi')' + 2\Lambda\phi g^{1/2}]. \tag{15}$$

That the conjugate momenta are completely determined by $g(x)$ and $\phi(x)$ reflects the absence of true degrees of freedom. It is straightforward to check that the constraints (14) and (15) imply in turn (4a) and (4b). Hence, one can take them as the basic constraints of the theory. Let us note at this point that, for a given C, not all configurations $g(x), \phi(x)$ of the spatial metric and the scalar field can be embedded in the classical (unique) solution of the field equations (1) and (3). Only those configurations which make P [Eq. (14)] real are classically admissible.

We are now in a position to solve the Wheeler-DeWitt equation, which is equivalent to

$$\delta\psi/\delta g(x) = -ig^{-1/2}[(\nabla\phi)^2 + \Lambda\phi^2 + C]^{1/2}\psi, \tag{16a}$$

$$\delta\psi/\delta\phi(x) = -i[(\nabla\phi)^2 + \Lambda\phi^2 + C]^{-1/2}[2\Delta\phi + 2\Lambda\phi]g^{1/2}\psi. \tag{16b}$$

I have set $\nabla\phi = g^{-1/2}\phi'$ and $\Delta\phi = g^{-1/2}(g^{-1/2}\phi')'$. $\nabla\phi$ and $\Delta\phi$ are spatial scalars.

An elementary line integral yields the unique general solution of (16) as

$$\psi[g(x), \phi(x)] = \exp\{iS[g(x), \phi(x)]\}, \tag{17}$$

where $S[g(x), \phi(x)]$ is the "Hamilton-Jacobi function"[17]

$$S[g(x),\phi(x)] = -2\int \phi g^{1/2}(\Delta\phi + \Lambda\phi)[(\nabla\phi)^2 + \Lambda\phi^2]^{-1}\{[(\nabla\phi)^2 + \Lambda\phi^2 + C]^{1/2} - C^{1/2}\}\,dx - 2C^{1/2}\int (g^{1/2} - 1)\,dx. \tag{18}$$

Although superficially noninvariant under spatial changes of coordinates because of the (infinite) constant $2C^{1/2}\int dx$ in S, the wave functional is actually invariant, for the integral $\int(dx - dx')$ vanishes for the coordinate transformations which preserve the boundary conditions [i.e., asymptotic translations ($\Lambda \geqslant 0$) or identity ($\Lambda < 0$)].

The solution (17) and (18) is the main result of this paper. One can eliminate the second derivative of ϕ from (18) by adding an appropriate divergence, but this is by no means necessary.

The solution of the Wheeler-DeWitt equation exhibits a certain number of interesting features. First of all, one sees that when the cosmological constant is zero (and with the values of C considered here), the exponent in the wave functional is real for all configurations $g(x), \phi(x)$ which obey the appropriate boundary conditions. Hence, these configurations are equally probable. This reflects the classical property that all these configurations can be embedded in the unique classical solution (5a) and (5b) of the field equations. Moreover, they all classically occur "an equal number of times." (This number is equal to the number of isometries which preserve the asymptotic conditions, i.e., to R^2.)

When the cosmological constant is positive, one has a similar situation, but the phase S of the wave functional is infinite. This phase can be made finite by the addition of an appropriate constant for a given value of α, but not for all of them. Now, α is a measure of time at infinity $[g \to \cosh^2(t/R) = \cosh^2\alpha]$. Hence, one sees that the wave functional is regular on the spacelike slices which approach a given slice at infinity, but not for those which asymptotically differ from it by a finite time translation. This seems to reflect the fact that $\partial/\partial t$ is not a Killing vector in the de Sitter case.

When the cosmological constant is negative, all the classically allowed configurations $g(x), \phi(x)$ have equal probability, while those which are classically forbidden—i.e., those which lead to an imaginary P when inserted in (14)—are exponentially damped (with the appropriate choice for the imaginary part of S).[18]

Lastly, I briefly indicate how the above results can be extended to the case of closed spatial sections. One finds then that the integration constant C cannot be frozen to a given value, but rather is an unconstrained degree of freedom with a well-defined momentum P_c. Both C and P_c have vanishing brackets with the constraints and yield a complete set of "observables,"[19] i.e., here, a complete set of gauge-invariant constants of the motion. In the quantum theory, one can diagonalize either $\hat{C}$ or $\hat{P}_c$, but not both. The solutions of the Wheeler-DeWitt equation which diagonalize $\hat{C}$ are similar to those described above. It is hoped to return to this question in the future.[20]

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17$S[g(x), \phi(x)]$ is determined up to an arbitrary constant chosen here in such a way that S is finite (this is only possible when $\Lambda \leq 0$; see text).

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