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Where Is the Continuum in Lattice Quantum Chromodynamics?

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We present a Monte Carlo calculation of the quark-liberating phase transition in lattice quantum chromodynamics. The transition temperature as a function of the lattice coupling g does not scale according to the perturbative beta function for $6/g^2 < 6.1$. We use finite-size scaling in analyzing the properties of the lattice system near the transition point.

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During the last few years gluon thermodynamics has been studied extensively within the framework of lattice quantum chromodynamics. Early strong-coupling calculations[1,2] and Monte Carlo simulation of the deconfining phase transition[3,4] opened the way to detailed quantitative results for the realistic SU(3) color group.[5]

We believe that the SU(3) deconfining phase transition with its expected first-order character[6] is a good laboratory to study the continuum limit of lattice QCD. The determination of the transition temperature is a unique test of the onset of scaling behavior as the continuum is approached. This expectation is based on the observation that T_c as a physical quantity is strictly nonperturbative and free from cutoff-dependent ultraviolet divergences which make string tension measurements difficult. Location of T_c is particularly easy because the system undergoes a sharp first-order phase transition where rounding effects are small.

The partition function for the Euclidean Wilson action $S_E(U)$ on the lattice is defined by the functional integral

$$Z = \int \prod_{\mathbf{x},t,\mu} dU \exp[-\beta S_E(U)], \tag{1}$$

where the integral in Eq. (1) is with respect to the Haar measure, and the inverse lattice coupling constant is $\beta = 6/g^2$. The Wilson action S_E is defined as a sum over all unoriented plaquettes:

$$S_E(U) = \sum_{\text{plaquettes}} (1 - \tfrac{1}{3}\,\mathrm{Re}\,\mathrm{Tr}\,UUUU). \tag{2}$$

We recall that the thermodynamics for finite temperatures is realized by lattices with spatial volume n_s^3 and temporal size n_t. The temperature T is $1/an_t$, where a is the lattice cutoff. Strictly speaking, n_s should be taken to infinity for fixed n_t in the thermo-

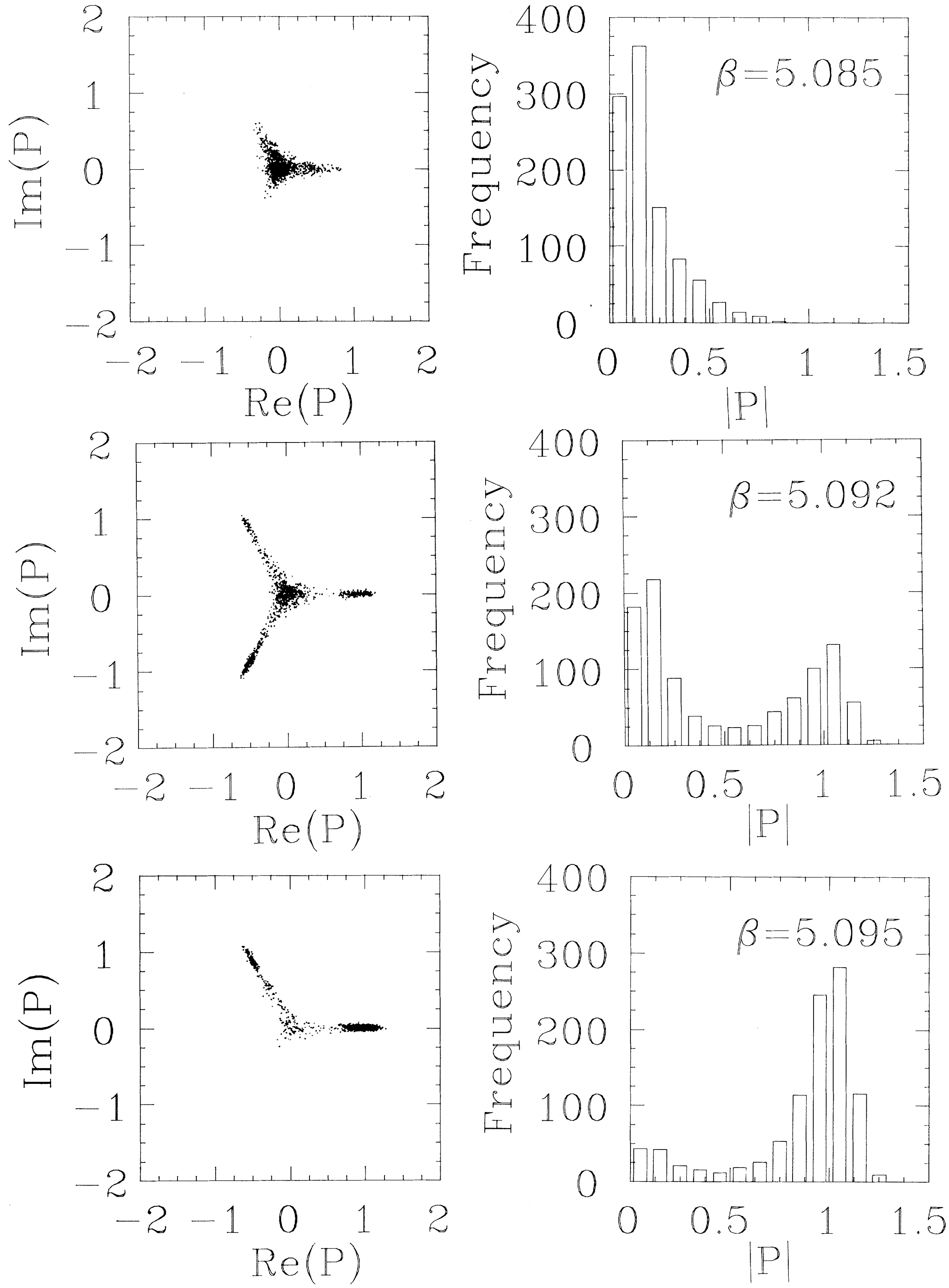

FIG. 1. The graphs on the left show the distribution of the Polyakov loop P in phase space on a 9^3 by 2 lattice. The top row corresponds to $\beta < \beta_c$ (confined), the middle row is at β_c, and the bottom row has $\beta > \beta_c$ (deconfined). Each point corresponds to the value of the average Polyakov loop on a given configuration, and each measurement is separated by 20 sweeps at β_c and 5 sweeps away from the transition region. The histograms on the right show the distribution of points as a function of the radial distance.

dynamic limit. This is only approximately realized in Monte Carlo calculations, and finite-size scaling becomes an important issue in the analysis.

The lattice spacing a is a known function of the coupling g in the continuum limit,

$$a = \Lambda_L^{-1}(16\pi^2/11g^2)^{51/121}\exp(-8\pi^2/11g^2), \quad (3)$$

where Λ_L is the lattice scale parameter. In this limit the scale can be set by $\Lambda_L = C\sqrt{\sigma}$ with the string tension $\sigma \approx (400\ \text{MeV})^2$. The constant C can be determined from Monte Carlo calculations of the static quark-antiquark potential.[7]

Following the method of Polónyi and Szlachányi,[8,9] we develop a qualitative physical picture for the phase transition which will guide our analysis of the Monte Carlo data. In strong coupling we can derive an effective action[10] in three dimensions from the partition function of Eq. (1) by integrating out the spatial link variables,

$$Z = \int [dP]\exp[-S_{\text{eff}}(P)]. \quad (4)$$

The effective action is

$$S_{\text{eff}}(P) = \beta_{\text{eff}}\sum_{\mathbf{x},i}|P(\mathbf{x}) - P(\mathbf{x}+\mathbf{e}_i)|^2 + \sum_{\mathbf{x}}\{-\tfrac{1}{2}\ln[27 - 18|P(\mathbf{x})|^2 + 8\,\text{Re}P(\mathbf{x})^3 - |P(\mathbf{x})|^4] - 6\beta_{\text{eff}}|P(\mathbf{x})|^2\}, \quad (5)$$

where $\beta_{\text{eff}} = (1/3g^2)^{n_t}$, and the Polyakov loop $P(\mathbf{x})$ is defined as

$$P(\mathbf{x}) \equiv \text{Tr}\prod_{t=1}^{n_t} U_t(\mathbf{x},t). \quad (6)$$

We shall denote the spatial average of $P(\mathbf{x})$ by P. S_{eff} is invariant under global Z_3 transformations. Mean-field calculations[10] predict a first-order phase transition at a value of β_{eff} above which the Z_3 symmetry is spontaneously broken. At the transition point the three broken Z_3 phases coexist with the unbroken phase at the origin.

Our Monte Carlo calculations were performed as follows: For each lattice size and value of β we generate gauge field configurations using the quasi-heat-bath method[11] of Cabibbo and Marinari. In order to execute the computation as efficiently as possible we use a lattice with helical boundary conditions. We measure the average value of the Polyakov loop P on these configurations, and plot the values on phase-space plots such as those of Fig. 1. This shows the results of three Monte Carlo runs: The first one with β below the transition point β_c illustrates the distribution found when the gluon plasma is in the confined phase; the second at β_c shows the case where the confined phase coexists with the deconfined phase; and the third above β_c has the points clustered around nonvanishing expectation values along the Z_3 directions, which is the signal for the deconfined phase.

The coexistence of the two phases over runs of great length (ten to forty thousand sweeps were typically required to study the system very close to the critical point) together with the jump in the order parameter P between the two phases provides conclusive evidence that the transition is of first order.

In order to measure β_c with sufficient accuracy to perform a finite-size scaling analysis we choose a precise quantitative criterion for determining the location of the transition point. We plot histograms, such as those in Fig. 1, showing the distribution of the magnitude of P and we choose β_c to be the value of β for which the number of points in the peak near the origin equals the number in the deconfined peak.

There is some arbitrariness in the definition of β_c for a given volume size n_s^3 depending on the location of the cut introduced between the two peaks. However, the arbitrariness in dividing the points into these two classes should not affect the value of β_c extrapolated to a lattice of infinite spatial volume. The width $\Delta\beta$, where the two phases coexist, of a temperature-driven first-order phase transition is expected to scale as[10,12]

$$\Delta\beta/\beta_c = \text{const}\times 1/sV, \quad (7)$$

where $V = n_s^3$ is the spatial volume and s is the latent entropy of the system.

Near a first-order phase transition we also anticipate

TABLE I. Values of the transition point β_c for lattices with n_t links in the temperature direction and n_s links in each spatial direction. The values extrapolated to lattices of infinite spatial extent are also given.

n_t	n_s	β_c
2	5	5.071 ±0.001
	7	5.086 ±0.002
	9	5.092 ±0.0015
	11	5.0945 ±0.001
	∞	5.097 ±0.001
4	7	5.669 ±0.005
	9	5.680 ±0.005
	11	5.690 ±0.003
	∞	5.696 ±0.004
6	7	5.830 ±0.01
	9	5.853 ±0.002
	11	5.865 ±0.004
	∞	5.877 ±0.006
8	7	5.95 ±0.03
	11	5.985 ±0.015
	∞	6.00 ±0.02
10	11	6.09 ±0.03

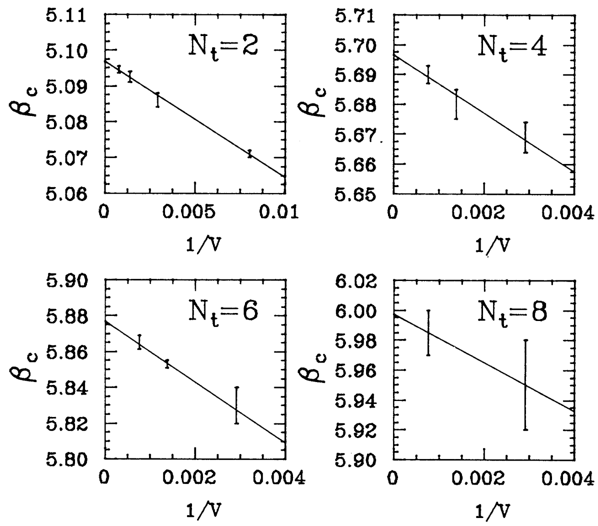

FIG. 2. Monte Carlo measurements of β_c for different spatial volumes at $n_t = 2$, 4, 6, and 8. The lines show the finite-size scaling extrapolations.

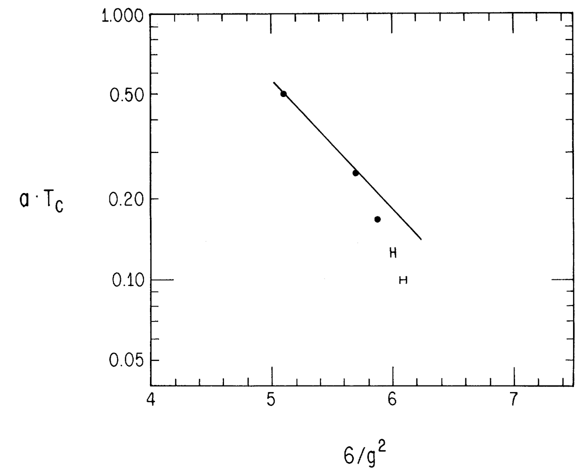

FIG. 3. The transition temperature aT_c as a function of the coupling constant. The solid line is the asymptotic scaling curve with $T_c = 77.5\Lambda_L$.

a shift in β_c as a function of the volume V which should scale as $1/V$.

Table I summarizes our results; in it are shown the values of β_c for several lattice sizes. In Fig. 2 we have plotted β_c as a function of the inverse volume for $n_t = 2$, 4, 6, and 8. The measured values lie on a finite-size scaling curve from which we can extrapolate our results to infinite volume at fixed n_t. The infinite-volume values for β_c thus obtained are also included in the table.

The errors quoted for β_c on a lattice with finite spatial volume give the range over which the measured distribution for the Polyakov loop is consistent (within statistical errors) with our criterion for β_c. Other choices for this criterion would lead to somewhat different finite-volume β_c values, but they should lie within the $1/V$ rounding of the phase transition. The errors on the infinite-volume values are determined from a linear fit by the $1/V$ scaling law. We conclude that finite-spatial-volume effects are under control in our analysis.

In Fig. 3 we show aT_c as a function of the coupling constant. The slope of $aT_c(g^2)$ is steeper by a factor of about 2 when compared with the two-loop renormalization-group beta function for $5.7 < 6/g^2 < 6.1$.

We have not extrapolated the point at $n_t = 10$ to infinite volume. The shift in β_c is expected to be < 0.03. Our conclusion is that aT_c does not obey perturbative scaling for $\beta < 6.09$. A detailed comparison of the scaling properties of T_c, the interquark potential, string tension,[7] and ratio test results[13] are given elsewhere.[14]

Our computations were performed on the CYBER 205 computer at the University of Karlsruhe. The details of our fully vectorized program are described elsewhere.[15] We have performed many checks on its reliability; in particular, the agreement with a high-order strong-coupling expansion for the Polyakov-loop correlation function at $\beta = 4$ and $n_t = 2$ is very satisfactory.

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