Tricritical Phenomena in Rotating Couette-Taylor Flow

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We present measurements of the fluid velocity in a rotating Couette-Taylor system of aspect ratio L near 1. At small angular speed ω of the inner cylinder, the system contains a symmetric vortex pair. As ω increases, the vortex boundary moves off center, and a suitably defined order parameter ψ becomes nonzero. This bifurcation changes from forward to backward as L is increased. Results for ψ agree quantitatively with the predictions of a Landau model for tricritical behavior.

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The close conceptual relationship between bifurcations in pattern-forming nonequilibrium systems and equilibrium phase transitions has long been appreciated, but, to our knowledge, only forward bifurcations, corresponding to ordinary critical points, have been studied experimentally by *quantitative* methods.[1] Considerable theoretical interest in more complicated systems has developed[2] recently, involving, for instance, backward or forward bifurcations to time-periodic or stationary states and codimension-two bifurcations. An interesting case arises when a forward bifurcation becomes backwards as a parameter is varied. This situation corresponds to the change from continuous transitions upon crossing a line of critical points to discontinuous ones associated with a line of first-order phase transitions, such as occurs at a tricritical point.

In this Letter, we report extensive, quantitative fluid velocity measurements near such a tricritical point in a fluid mechanical system driven far from equilibrium by an external stress. Our data can be fitted very accurately by a Landau model[3] equivalent to that used near symmetric equilibrium tricritical points.[4]

The physical system is rotating Couette-Taylor flow between two coaxial cylinders with the inner one rotating at an angular speed ω. The active section of the apparatus is so short that there is only one vortex pair. This means that the aspect ratio L (the ratio of the length H to the width d of the gap between the cylinders) is about 1. If ω is smaller than a critical value ω_1, the vortices are symmetric, and the boundary between them is in the center (assuming the apparatus is "perfect"). If the speed is greater than ω_1, one vortex starts to grow at the expense of the other. This is illustrated by the data for the axial velocity component v_z which are shown in Fig. 1 as a function of axial position for various ω. The solid circles correspond to the symmetric state with $\omega < \omega_1$, and they give $v_z = 0$ at the geometric center of the system. When $\omega > \omega_1$, the zero crossing of v_z can move either to the right (solid symbols) or to the left (open symbols). In either case, the symmetry is broken and a bifurcation (phase transition) has taken place.

For a perfect system, the symmetry is broken at $\omega = \omega_1$, either smoothly, with the zero crossing for v_z being displaced from the center continuously as ω grows beyond ω_1, or abruptly. The former case corresponds to a forward bifurcation, or a second-order transition, and occurs for relatively small aspect ratios $L \leq L_t$ with $L_t = 1.255$ for our system. The latter corresponds to a backward bifurcation, or first-order transition, and is observed for $L_t < L < L_c$ where $L_c = 1.292$ for our experiment. This phenomenon has been studied previously by Benjamin and Mullin,[5] Cliffe,[6] Lücke *et al.*,[7] and Schmidt,[8] but, to our knowledge, there have been no systematic, quantitative measurements of an appropriate order parameter which can be compared, for instance, with the Landau theory of behavior near a tricritical point.

In our apparatus the inner cylinder radius was $r_1 = 1.737$ cm, and the radius ratio $\eta \equiv r_1/r_2$ (r_2 is the outer cylinder radius) was 0.500. The fluid was confined between nonrotating axial boundaries whose separation H was adjustable. The aspect ratio $L = H/d$

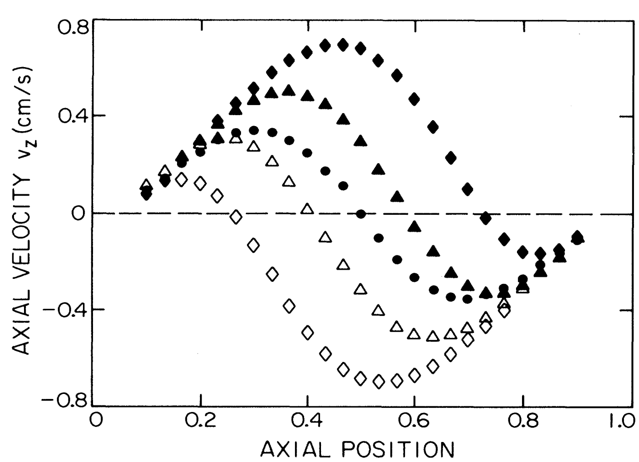

FIG. 1. The axial velocity v_z for various angular velocities ω_1 as a function of axial position. The scale of the abscissa is normalized by the active apparatus length H. The measurements are for an aspect ratio of 1.129. Circles: $\omega = 2.56\ \mathrm{s}^{-1}$; triangles: $\omega = 2.86\ \mathrm{s}^{-1}$; lozenges: $\omega = 3.28\ \mathrm{s}^{-1}$. Solid symbols correspond to the branch which is reached by quasistatically increasing ω in our apparatus. For this run, ω_1 (see below) was $2.794\ \mathrm{s}^{-1}$.

was varied in a narrow range near 1 by changing H and was measured with an accuracy of 0.2%. The column was horizontally mounted. The inner cylinder's angular speed was controlled accurately by a frequency synthesizer. The fluid was a 50% (by volume) solution of glycerol in water. Its temperature was 21.0 °C and constant to within $\pm 5\times 10^{-3}$ °C. Its kinematic viscosity as determined with an Ostwald viscometer was 0.0661 cm^2/s. The fluid velocity was measured using laser Doppler velocimetry (LDV). For this purpose, the fluid was seeded with 1.2 $\mu g/cm^3$ of polystyrene-latex spheres of diameter 1.09 μm. The velocimeter was mounted on a motor-driven stage and could be moved with a resolution of 1.3×10^{-3} cm. We measured the axial velocity component at a radial distance of about 0.24 cm from the inner cylinder at 25 points equally and symmetrically spaced along the cylinder axis. The separation of these points was $H/30$. After completing all of these measurements, we increased L to the maximum possible value for our apparatus ($L=15$) without changing the fluid and determined $\omega_c = 1.511$ s^{-1} for the onset of Taylor vortex flow in the infinite system.[9] This corresponds to a critical Reynolds number $R_c \equiv r_1\omega_c d/\nu = 69.0$, which compares well with the theoretical value[10] $R_c = 68.2$.

In analogy with the definition used by Lücke *et al.*,[7] we define an order parameter by

$$\psi = \int_0^H v_z dz \Big/ \int_0^H |v_z| dz. \qquad (1)$$

For a perfect bifurcation, we expect that $\psi = 0$ for $\omega \leq \omega_1$ and $-1 < \psi < 1$ for $\omega > \omega_1$. In practice, we replace the right-hand side of Eq. (1) by the ratio of the appropriate sums over our data points. Results for ψ obtained at the three aspect ratios $L = 1.129$, 1.266, and 1.281 are shown in Fig. 2 as a function of

$$\epsilon \equiv \omega/\omega_1 - 1. \qquad (2)$$

Concentrating on Fig. 2(a), we see that the bifurcation is imperfect [$\psi(\epsilon)$ appears "rounded" near $\epsilon = 0$]. Quasistatic increases of ω always resulted in a vortex-boundary displacement in the same direction, corresponding to the upper, favored branch. However, by appropriate manipulation of the movable boundary of the system, followed by an increase of ω from $\omega < \omega_1$ to $\omega > \omega_1$, it was possible to prepare a state corresponding to the unfavored, lower branch. Measurements of $\psi(\epsilon)$ were then possible with ϵ decreasing. For ϵ near zero, that state ceased to exist, and a transition to $\psi > 0$ on the favored branch occurred, as illustrated by the solid lozenges in Fig. 2(a). There the data agree extremely well with the data obtained by increasing ϵ quasistatically from $\epsilon < 0$ (open circles).

To the data in Fig. 2(a) we fitted a Landau model with a "free energy" F given by

$$F = -h\psi - \tfrac{1}{2}\epsilon\psi^2 + \frac{g}{4}\psi^4 + \frac{k}{6}\psi^6. \qquad (3)$$

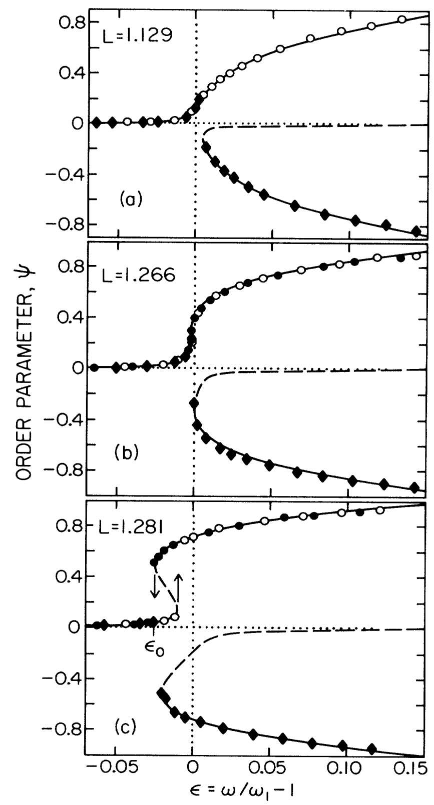

FIG. 2. Measured values of the order parameter ψ [see Eq. (1)] for various values of ϵ for three aspect ratios, L. Data taken by quasistatically increasing ϵ are shown as open symbols, while solid symbols show data obtained by decreasing ϵ. The curves are the result of a fit by a Landau model [see Eqs. (3) and (4)]. The solid portions are the stable branches (minima in the "free energy"), and the dashed portions show the unstable branches (maxima in the "free energy"). In all cases, the upper branch is favored and is always followed when ϵ is increased quasistatically from negative values. (a) shows data for L well below the tricritical value ($L_t = 1.255$), and (b) corresponds to L only very slightly above L_t. Note the qualitatively different behavior near $\epsilon = 0$, which is discussed in the text. (c) is for L sufficiently greater than L_t to observe the discontinuous transitions resulting from the backward bifurcation, which are indicated by arrows.

In this equation, we include even terms in ψ because *a priori* for the perfect system we expect F to be independent of the sign of ψ. Since the *experiment* reveals a "rounded" transition to a *favored* branch, we include also the lowest-order asymmetric term $h\psi$. In the thermodynamic analogy, h plays the role of a field conjugate to ψ. In our system, this field presumably is caused by imperfections in the apparatus. For given values of the parameters h, ϵ, g, and k, ψ can assume any value corresponding to a minimum of F, and, at the extrema of F, $\psi(\epsilon)$ is given by

$$h + \epsilon\psi - g\psi^3 - k\psi^5 = 0. \quad (4)$$

For $h=0$, it is easy to find the roots of Eq. (4) analytically. For *positive* g, one obtains $\psi \sim \epsilon^{1/2}$ for small ϵ, corresponding to a forward bifurcation. This square-root behavior is qualitatively evident from an inspection of Fig. 2(a). Indeed, a least-squares fit of Eqs. (2) and (4), adjusting ω_1, h, g, and k, yielded $g>0$. However, as L increases, the parameters resulting from such a fit change. Results for g and ω_1/ω_c are given in Fig. 3. One sees that g [Fig. 3(a)] decreases with increasing L, passing smoothly through zero for $L \equiv L_t = 1.255$. The case $h=g=0$ corresponds to a tricritical point at $\epsilon=0$, and near there Eq. (4) yields $\psi \sim \epsilon^{1/4}$ for small ϵ. Data for L near L_t are shown in Fig. 2(b). The rapid variation of ψ with ϵ near $\epsilon=0$, corresponding closely to $\epsilon^{1/4}$, is noticeable, particularly by a comparison with Fig. 2(a).

As L is increased beyond L_t, g becomes negative and Eq. (4) with $h=0$ predicts that the transition becomes first order (the bifurcation is backward). This phenomenon is illustrated by the data in Fig. 2(c). In this case, the open circles were obtained with increasing ϵ. For negative ϵ close to zero, ψ is already positive (although small) in the presence of the field h. Before $\epsilon=0$ is reached, a discontinuous jump to $\psi \simeq 0.65$ occurs. This is indicated by the upward pointing arrow in Fig. 2(c). With decreasing ϵ (solid circles), a hysteresis loop is swept out, and a jump to ψ near zero occurs at a significantly negative ϵ as indicated by the downward pointing arrow in Fig. 2(c). As in Fig. 2(a), the curves in Figs. 2(b) and 2(c) correspond to the fits of Eqs. (2) and (4) to the data. The solid portions of the curves show stable solutions (minima of F), and the dashed sections show unstable solutions (maxima of F). The fit to the data is good in all cases.

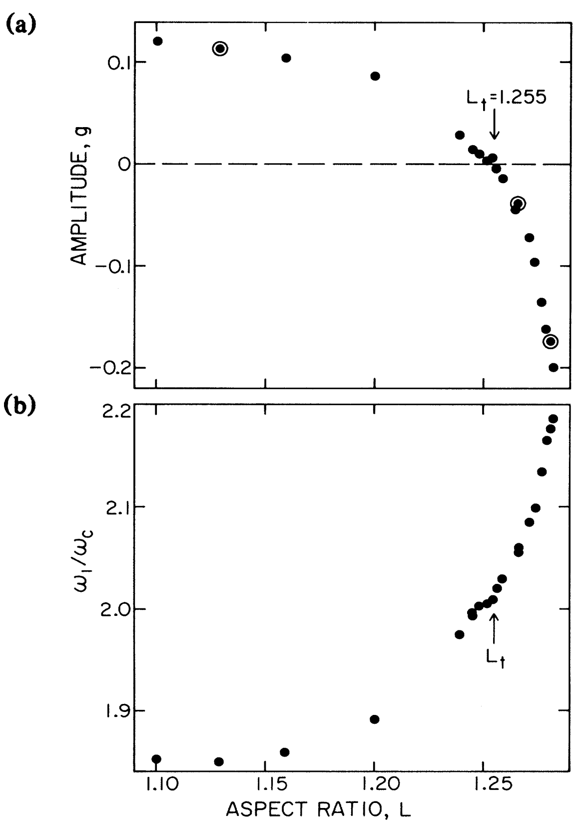

FIG. 3. The parameters g and ω_1 obtained by fitting Eqs. (2) and (4) to the data as functions of the aspect ratio L. The data for ω_1 have been normalized by ω_c, the critical angular velocity for the onset of Taylor vortex flow in a system of infinite L. The open circles in (a) correspond to the data shown in Fig. 2.

If h were negligible, the data of Fig. 2(b) would presumably show hysteresis, since, for $L=1.266$, g is negative (-0.039). Indeed, for $h=0$ the width ϵ_0 [see Fig. 2(c)] of the hysteresis loop given by Eq. (4) is equal to $-g^2/4k$. The fits, which give $h \simeq 8\times10^{-4}$ near $L=L_t$, show no hysteresis, however, for $g \gtrsim -0.06$. In fact the field h has the effect of increasing the value of L for which hysteresis is first observable from L_t to 1.270.

It is interesting to note that the fits to the data for L in the range of our experiment yield $5\times10^{-4} \lesssim h \lesssim 10^{-3}$. Thus the field is extremely small (i.e., the experimental apparatus is "nearly perfect"). The relatively large effect of h upon ψ near $\epsilon=0$ is associated with the divergent susceptibility at $\epsilon=0$. Indeed, it is easy to see from Eq. (4) that $\psi(\epsilon=0) \sim (h/g)^{1/3}$ for positive g and $\psi(\epsilon=0) \sim (h/k)^{1/5}$ for $g=0$. Thus ψ can become appreciable even for small h. It is clear that improving the apparatus by, say, an order of magnitude will reduce the "rounding" near $\epsilon=0$ by only a factor of 2 or so.

Our results for ω_1/ω_c are summarized in Fig. 3(b). They agree quite well with the recent data of Schmidt,[8] who used the same radius ratio $\eta=0.5$ as ours. The results for the parameter k [see Eq. (3)] have not been shown, but their dependence on L is very similar to that of ω_1/ω_c, with k increasing smoothly from 0.15 at $L=1.20$ to 0.33 at $L=1.28$.

After this work was completed, we became aware of calculations of the phase diagram from numerical solutions of the Navier-Stokes equations.[11] Those results yield $L_t=1.259$, only 0.3% larger than our experimental value. The computed $\omega_1(L)$ differs by less than 1% from the measured values in Fig. 3 (most of the small difference could be removed by a shift along the abscissa so as to cause the results for L_t to agree exactly). Good agreement also exists for the width ϵ_0 of the hysteresis loop for $L > L_t$.

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[1]See, for instance, J. P. Gollub and H. Freilich, Phys. Fluids **19**, 618 (1976) or J. Wesfreid, Y. Pomeau, M. Dubois, C. Normand, and P. Bergé, J. Phys. (Paris) **39**, 725 (1978), or G. Pfister and I. Rehberg, Phys. Lett. **83A**, 19 (1981).

[2]See, for instance, V. Steinberg and H. R. Brand, Phys. Rev. A **30**, 3366 (1984), and references therein.

[3]L. Landau, Phys. Z. Sowjetunion **11**, 26 (1937), reprinted in *Collected Papers of L. D. Landau,* edited by D. ter Haar (Pergamon, London, 1965), p. 193.

[4]R. B. Griffiths, Phys. Rev. Lett. **24**, 715 (1970).

[5]T. B. Benjamin and T. Mullin, Proc. Roy. Soc. London, Ser. A **377**, 221 (1981).

[6]K. A. Cliffe, J. Fluid Mech. **135**, 219 (1983).

[7]M. Lücke, M. Mihelcic, K. Wingerath, and G. Pfister, J. Fluid Mech. **140**, 343 (1984).

[8]H. Schmidt, thesis, University of Kiel, West Germany, 1983 (unpublished).

[9]The measurements of ω_c were extrapolated to infinite L in a manner similar to that used by Pfister and Rehberg (Ref. 1), and the result for ω_c is believed to differ from the value for infinite L by no more than 0.1%.

[10]M. A. Dominguez-Lerma, G. Ahlers, and D. S. Cannell, Phys. Fluids **27**, 856 (1984).

[11]K. A. Cliffe, private communication.