Accelerated Diffusion in Josephson Junctions and Related Chaotic Systems

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(Received 29 October 1984)

We report a new type of anomalous diffusion in chaotic systems with periodic symmetry. It is characterized by mean square displacements diverging faster than linearly in time. By a comparison of power spectra we conclude that the phenomenon has occurred in a recent observation of $1/f$ noise in Josephson junctions.

PACS numbers: 05.45.+b, 66.30.–h, 74.50.+r

When a particle diffuses, its mean square displacement usually grows linearly in time (for $t \to \infty$). Deviations from this linear growth, known as anomalous diffusion, were found in amorphous solids,[1] superionic conductors,[2] polymer melts,[3] and other systems[4] and have stimulated the theoretical work.[5] Anomalous diffusion was also predicted to show up in certain chaotic systems.[6] It is well known by now that resistively shunted Josephson junctions behave chaotically if they are driven by a periodic current.[7] This is also the case for dc biased junctions, if the shunt includes a substantial self-inductance.[8] Recently Miracky, Devoret, and Clarke (MDC) have reported the observation of $1/f$ noise in the voltage power spectrum of such a junction[9] (Josephson analog). We will argue below that this observation is associated with a new type of anomalous diffusion in chaotic systems: While in all above cases[1–5] including the chaotic systems of Ref. 6 the mean square displacement $\sigma^2(t)$ grows slower than linearly (i.e., like t^α with $\alpha < 1$), here it grows faster than linearly ($\alpha > 1$). Diffusion thus is not inhibited but accelerated as compared with normal diffusion ($\alpha = 1$).[10] The diffusion coefficient $D = \lim[\sigma^2(t)/2t]$ does not vanish as in Ref. 6 but diverges. We describe the theory in some generality and later discuss its application to Josephson junctions. Depending on a universality exponent z the mean square displacement generally grows like $t^{3-1/(z-1)}$, like t^2, or like $t\ln t$. For comparison with experiments we also determine the corresponding power spectra.

We consider a class of systems having discrete translational symmetry like, e.g., a driven damped particle in a periodic potential $V(x) = V(x+n)$. The periodicity defines unit cells of length 1 along the x axis; diffusion may arise as a random motion from cell to cell. This example is an analog for the phase dynamics of resistively shunted Josephson junctions[7] (where x stands for the phase ϕ) and for a phenomenological model of weakly pinned charge-density waves.[11] For sufficient damping the chaotic dynamics of these systems can be approximately described by circle maps,[12–15]

$$x_{t+1} = f(x_t), \quad f(x+n) = f(x) + n, \tag{1}$$

where the second equation expresses discrete translational symmetry and f is a map whose actual form depends on the particular physical problem and its parameters. We require that it has at least one maximum per unit cell and for convenience assume reflection symmetry[16]:

$$f(-x) = -f(x). \tag{2}$$

An example is depicted in Fig. 1(a). Maps of this type can explain a number of phenomena reported for Josephson junctions.[17,18]

The systems described by Eqs. (1) and (2) exhibit chaotic motions in the form of diffusion (phase diffusion for Josephson junctions).[12–14] In the situations studied so far, diffusion was associated with no anomalous spectra[12,17,18] or with spectra increasing like ω^β (with $\beta > 0$).[6,19] Here we present a general mechanism, which can cause an accelerated diffusion process and is associated with $1/f$ spectra: Variations in the physical parameters of a system cause variations in the shape of the corresponding map. If thereby the map exhibits intermittency[20,21] in two transfer regions (i.e., where transfers to neighboring cells take place), the following theory [Eqs. (4)–(11)] applies. For the sake of clarity we illustrate our ideas in a particular example, Fig. 1. When the maximum of a map (or of its

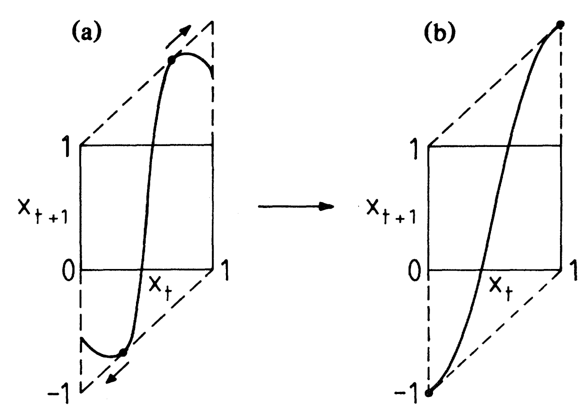

FIG. 1. (a) Example of a map satisfying Eqs. (1)–(3) shown for a unit cell $0 \le x_t \le 1$. Solid circles are contact points with the lines $\pm 1 + x_t$. (b) A suitable change of parameters may cause the contact points to move to the cell boundary.

iterates) moves up or down, it can become tangent to the (dashed) line $x_{t+1}=1+x_t$ in Fig. 1(a). This situation was frequently encountered with Josephson junctions.[18,22] One can now imagine a variation where the contact points move to the cell boundaries as in Fig. 1(b). In their vicinity ($x \approx m$) the map then is of the form

$$f(x)=(1+\epsilon)(x-m)+a(x-m)^z+m-1 \quad (m \lesssim x < m+\tfrac{1}{2}), \tag{3}$$

where $x-m << 1$ and $\epsilon \to 0$. As usual for intermittent systems[21,23] we allow for a general exponent $z>1$ distinguishing universality classes; $z=2$ is the generic case. In order to avoid problems associated with the lack of an invariant measure (for $z \geqslant 2$) we have introduced the parameter ϵ and let $\epsilon \to 0$. Aside from the limiting form Eq. (3) and the symmetry conditions Eqs. (1) and (2), the map need not be specified. We only require a smooth injection probability to the vicinity of $x=m$.

When an orbit x_t reaches the vicinity of a point $x=m$, according to Fig. 1(b) and Eq. (3) it is transferred to an almost equivalent position in the neighboring cell. This transfer is repeated successively, resulting in a laminar motion of correlated jumps over many cells. We will give a brief outline of our theory; more details will be presented elsewhere.[24] Let T denote the duration of laminar phases. Its probability density $\psi(T)$ can be calculated in a continuous-time approximation for intermittent systems,[21,23] which here yields

$$\psi(T)=2a^{-\nu}\epsilon^{\nu+1}[(1+2^{z-1}\epsilon/a)e^{\epsilon(z-1)T}-1]^{-\nu}+2a^{1-\nu z}\epsilon^{\nu z}[(1+2^{z-1}\epsilon/a)e^{\epsilon(z-1)T}-1]^{-\nu z}, \tag{4}$$

where $\nu=1/(z-1)$. The velocity autocorrelation function $C(t)=\langle v_0 v_t\rangle$ can be obtained with the help of renewal theory[25] or by phase-space averages,

$$C(t)=\frac{1}{\langle T\rangle}\int_t^\infty (T-t)\psi(T)\,dT. \tag{5}$$

Depending on the exponent z we had to use different routes for the following calculations. For $z \neq 2$ (and $\epsilon=0$ for $z<2$) the Laplace transform $\tilde{\psi}(s)$ of $\psi(T)$ exists and Eq. (5) can be written in Laplace transform,

$$\tilde{C}(s)=s^{-1}+\langle T\rangle^{-1}s^{-2}[\tilde{\psi}(s)-1]. \tag{6}$$

The mean square displacement $\sigma^2(t)=\langle (x_t-x_0)^2\rangle$ follows from Eq. (6) as

$$\sigma^2(t)=L^{-1}\{2s^{-3}+2[\lim_{s'\to 0} d\tilde{\psi}/ds']^{-1}s^{-4}[1-\tilde{\psi}(s)]\}, \tag{7}$$

where L^{-1} denotes the inverse Laplace transform. For $z=2$ we had to calculate the correlation function explicitly by integrating Eq. (5):

$$C(t)=-[\ln(1+a/2\epsilon)]^{-1}\ln[1-ae^{-\epsilon t}/(a+2\epsilon)], \tag{8}$$

which in the limit $\epsilon \to 0$ and $s/\epsilon >> 1$ yields

$$\tilde{C}(s)=[\ln(a/2\epsilon)]^{-1}[\gamma s^{-1}+s^{-1}\ln(s/\epsilon)], \tag{9}$$

where γ is Euler's constant. From Eqs. (4) and (7) or Eq. (9) we have calculated the mean square displacements in the long-time limit,[24] making use of Tauberian theorems and of Karamata's theorem. For $\epsilon \to 0$ and with the ommision of nonuniversal prefactors they are

$$\sigma^2(t) \sim \begin{cases} t^2, & z \geqslant 2, \\ t^{3-1/(z-1)}, & \frac{3}{2}<z<2, \\ t\ln t, & z=\frac{3}{2}, \\ t, & 1<z<\frac{3}{2}. \end{cases} \tag{10}$$

We illustrate these results by a computer simulation in Fig. 2, where $\sigma^2(t)$ has been determined by averaging over 2000 orbits of maps belonging to Eqs. (1)–(3). For $z>\frac{3}{2}$ the asymptotic growth t^α is anomalous with exponents $\alpha>1$. Diffusion is thus enhanced as compared with normal diffusion (t^1). For $z \geqslant 2$ it attains the strongest possible power law t^2. This should not be misinterpreted as a steady drift motion [where $(x_t-x_0)^2=v^2t^2$]. The process is indeed a random walk with characteristic power spectra (see below) distinguishing it from other random walks. As a consequence of the enhancement of diffusion ($\alpha>1$) the diffusion coefficient $D=\lim[\sigma^2(t)/2t]$ (for $t\to\infty$) diverges, whereas in other cases of anomalous diffusion[1-6] it vanishes.

In experiments the above phenomenon will manifest itself through its power spectra. Although they are not meant to be the main message of this paper, below we give the velocity power spectra to allow comparison with experiment. They follow as the Fourier trans-

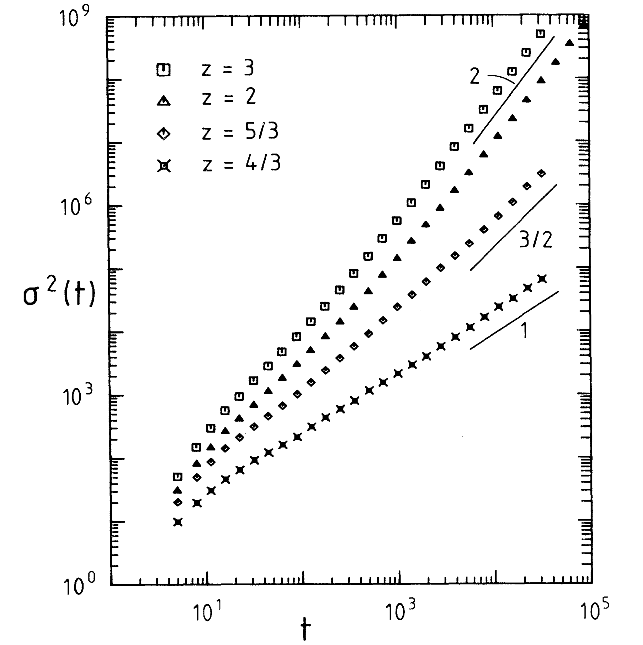

FIG. 2. Computer simulation illustrating the anomalous asymptotic growth of the mean square displacement Eq. (10) in a log-log plot. Theoretically predicted slopes are indicated by straight lines. The anomalous growth (for $z \geq \frac{3}{2}$) is accelerated as compared with normal diffusion ($z = \frac{4}{3}$).

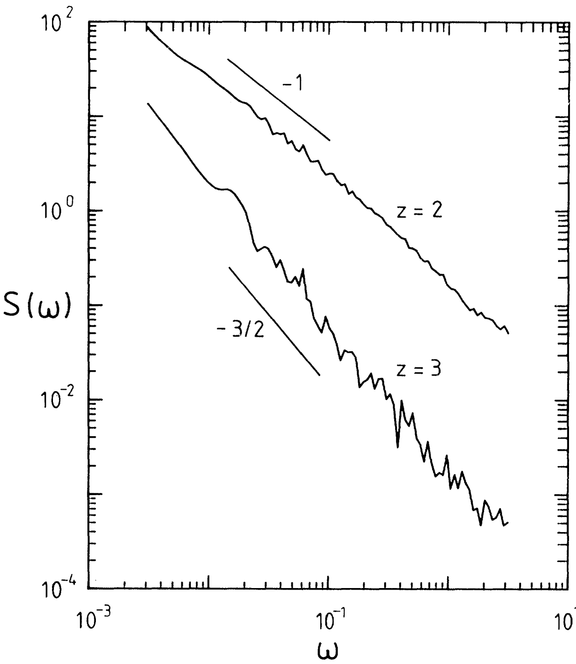

FIG. 3. Velocity power spectrum $S(\omega)$ exhibiting $1/f$ noise in a computer simulation for $z=2$ and $z=3$. Straight lines indicate the analytic result Eq. (11). In a Josephson experiment this spectrum is the *voltage* power spectrum.

forms of Eqs. (5) and (8), i.e., $S(\omega) = 2\,\mathrm{Re}\,\tilde{C}(\delta - i\omega)$ with $\delta \to 0$. Omitting nonuniversal prefactors we obtain

$$S(\omega) \sim \begin{cases} \omega^{-1}(\epsilon/\omega)^{1-1/(z-1)}, & z > 2, \\ -(\pi/\ln\epsilon)\omega^{-1}, & z = 2, \\ \omega^{-2+1/(z-1)}, & \frac{3}{2} < z < 2, \\ |\ln\omega|, & z = \frac{3}{2}, \\ \mathrm{const}, & 1 < z < \frac{3}{2}. \end{cases} \tag{11}$$

Here we had to consider the regime $\epsilon \ll \omega \ll \pi$, i.e., $\omega/\epsilon \gg 1$. Note that this is the only means to arrive properly at the $1/f$ noise in the cases $z \geq 2$, where the $\omega^{-\gamma}$ behavior (with $\gamma \geq 1$) cannot extend to $\omega = 0$. The divergence of the integral $\int \omega^{-\gamma}\, d\omega$ would otherwise be incompatible with a well-behaved correlation function $C(t=0)$. For $\frac{3}{2} < z < 2$ the decay is still like $\omega^{-\gamma}$, but with $\gamma < 1$. Figure 3 illustrates the $1/f$ noise by a computer simulation for two maps satisfying Eqs. (1)–(3). Note that the spectra for $z \geq 2$ are markedly different from those of other intermittent systems.[19,20]

We finally discuss the application of the theory to Josephson junctions and MDC's observation.[9] It cannot explain other observations of $1/f$ noise in Josephson junctions. By adding external noise MDC could show that their $1/f$ noise was a deterministic (i.e., chaotic) phenomenon occurring for a special choice of parameters. They were led to study an idealized *voltage* return map, which does not shed light on diffusive motions. By consideration of the *phase* ϕ instead of the *voltage* as dynamical variable, their observation appears as a manifestation of accelerated diffusion (of the phase): Its velocity $v = \dot{\phi} = U2e/\hbar$ is proportional to the voltage U across the junction. Generally there is a close connection [used in Eq. (7)] between mean square displacements and velocity power spectra. The observation[9] of ω^{-1} noise in the voltage (i.e., velocity) power spectrum thus is connected with anomalous mean square displacements of the phase ϕ. We have presented a general mechanism causing such anomalous deterministic diffusion. Since for $z=2$ it yields the observed spectral shape (pure ω^{-1}) we believe it was operative, although details may be more complex (the actual width of the ω^{-1} regime depends on ϵ). The junction's highly nonlinear equation of motion prevents an analytic derivation of our model. For symmetry reasons the map must be a perturbed circle map, Eqs. (1) and (2).[12–15] A limiting form like Eq. (3) may be sought in analog computer simulations.

This work was supported in part by the Deutsche Forschungsgemeinschaft.

[1]M. E. Scharfe, Phys. Rev. B **2**, 5025 (1970); H. Scher and E. W. Montroll, Phys. Rev. B **12**, 2455 (1975).

[2]J. Bernasconi, H. U. Beyeler, S. Strässler, and S. Alexander, Phys. Rev. Lett. **42**, 819 (1979).

[3]D. Richter *et al.*, Phys. Rev. Lett. **47**, 109 (1981).

[4]E.g., R. Kutner and K. W. Kehr, Philos. Mag. A **48**, 199 (1983); Y. Imry and B. Gavish, J. Chem. Phys. **61**, 1554 (1974); C. H. Bennet, M. Büttiker, R. Landauer, and H. Thomas, J. Stat. Phys. **24**, 419 (1981).

[5]For a review see, e.g., E. W. Montroll and M. F. Shlesinger, in *Studies in Statistical Mechanics,* edited by E. W. Montroll and J. L. Lebowitz (North-Holland, Amsterdam, 1984), Vol. 11, p. 1.

[6]T. Geisel and S. Thomae, Phys. Rev. Lett. **52**, 1936 (1984).

[7]B. A. Huberman, J. P. Crutchfield, and N. H. Packard, Appl. Phys. Lett. **37**, 750 (1980); E. Ben-Jacob, Y. Braiman, R. Shainsky, and Y. Imry, Appl. Phys. Lett. **38**, 822 (1981); N. F. Pederson and A. Davidson, Appl. Phys. Lett. **39**, 830 (1981); R. C. Kautz, J. Appl. Phys. **52**, 6241 (1981); E. Ben-Jacob, I. Goldhirsch, Y. Imry, and S. Fishman, Phys. Rev. Lett. **49**, 1599 (1982); W. J. Yeh and Y. H. Kao, Phys. Rev. Lett. **49**, 1888 (1982).

[8]R. F. Miracky, J. Clarke, and R. H. Koch, Phys. Rev. Lett. **50**, 856 (1983).

[9]R. F. Miracky, M. H. Devoret, and J. Clarke, Bull. Am. Phys. Soc. **29**, 481 (1984), and private communication, and to be published.

[10]Anomalous diffusion with $\alpha > 1$ was also reported in T. Schneider and E. Stoll, Phys. Rev. Lett. **41**, 1429 (1978), but was later amended to $t^{2/3}$ [Phys. Rev. B **22**, 395 (1980)].

[11]G. Grüner, A. Zawadowski, and P. M. Chaikin, Phys. Rev. Lett. **46**, 511 (1981); A. Zettl and G. Grüner, Solid State Commun. **46**, 501 (1983); P. Monceau, J. Richard, and M. Renard, Phys. Rev. Lett. **45**, 43 (1980); M. Sherwin, R. Hall, and A. Zettl, Phys. Rev. Lett. **53**, 1387 (1984).

[12]T. Geisel and J. Nierwetberg, Phys. Rev. Lett. **48**, 7 (1982).

[13]M. Schell, S. Fraser, and R. Kapral, Phys. Rev. A **26**, 504 (1982).

[14]H. Fujisaka and S. Grossmann, Z. Phys. B **48**, 261 (1982).

[15]M. H. Jensen, P. Bak, and T. Bohr, Phys. Rev. Lett. **50**, 1637 (1983); the relevance of circle maps is studied especially in P. Bak, T. Bohr, M. H. Jensen, and P. V. Christiansen, to be published.

[16]Giving up this symmetry results in additional drifting motions superimposed on diffusion (Ref. 5).

[17]T. Geisel and J. Nierwetberg, Phys. Rev. A **29**, 2305 (1984).

[18]See note added in proof to T. Geisel and J. Nierwetberg, Z. Phys. B **56**, 59 (1984).

[19]T. Geisel, unpublished.

[20]P. Manneville, J. Phys. (Paris) **41**, 1235 (1980).

[21]I. Procaccia and H. Schuster, Phys. Rev. A **28**, 1210 (1983).

[22]D.-R. He, W. J. Yeh, and Y. H. Kao, to be published.

[23]J. E. Hirsch, B. A. Huberman, and D. J. Scalapino, Phys. Rev. A **25**, 519 (1982).

[24]A. Zacherl, T. Geisel, and J. Nierwetberg, to be published.

[25]E.g., D. R. Cox and H. D. Miller, *The Theory of Stochastic Processes* (Methuen, London, 1965).