Invariants Polynomial in Momenta for Integrable Hamiltonians

Laurence S. Hall
Lawrence Livermore National Laboratory, Livermore, California 94550
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We show the near-universal existence of a second invariant that is polynomial in the momenta for integrable Hamiltonian systems in two dimensions. Specifically, Hietarinta's three "counterexamples" are converted to polynomial form.

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There has been much interest recently in the theory of integrable Hamiltonian systems in two dimensions. I have previously argued that the existence of an independent second invariant of polynomial form in the momenta is implied by the existence of an invariant of more general form.[1] This proposition is of considerable importance, at least for the physically most-significant class of Hamiltonians

$$H = \tfrac{1}{2}(p_x - A_x)^2 + \tfrac{1}{2}(p_y - A_y)^2 + V, \tag{1}$$

where A_x, A_y, and V depend upon position in configuration space. Then, complete procedures for the determination of the polynomial invariant can be written down.[2]

The proof of my proposition as given in Ref. 1 is imperfect; attention was not directed to ensuring nontriviality of the ultimate polynomial form. Indeed, Hietarinta has argued the opposite side.[3] He studies three specific Hamiltonians and concludes that for these three cases, the second invariant is fundamentally *rational*, an *elementary transcendental*, or a *higher transcendental* function in momenta.

We shall show that Hietarinta's counterexamples are narrowly defined; too narrowly, we argue. Polynomial invariants exist in each case.

A variety of terminology has been applied to the classes of functions that remain constant under Hamiltonian flow; these functions are variously called invariants, integral invariants, integrals, first integrals, integral constants, constants of the motion, etc.

Consistent with Whittaker[4] and with Abraham and Marsden,[5] we may consider *integrals* or *first integrals* or *integral invariants* or *constants of the motion* to be functions defined on an (open) phase-space manifold M; we use the term *integral constants* for the values of the arbitrary constants that specify the integral invariants. In general, specific values of the integral constants define a submanifold of M.

More inclusively, we may call any function $J(p,q,t)$ an invariant if $dJ/dt = 0$ for whatever specific flows are in question. Such flows, depending upon the context, may be appropriate only to a submanifold of M, or to M as a whole, or they even may be associated with a pertinent extended manifold. In Whittaker's terminology,[4] the equation which assigns a not-necessarily arbitrary value to J, e.g., $J(p,q,t) = 0$ on M, is an *invariant relation.*

We have previously used the term *configurational invariant* in a discussion of both exact and approximate invariants to emphasize the macroscopic origin of an invariant J, as opposed to the microscopic origin of an adiabatic invariant.[1] However, the configurational invariants were introduced in the context of flows possibly restricted to a submanifold of M. Subsequently, Sarlet, Leach, and Cantrijn[6] have insisted, quite correctly, that a clear distinction be maintained between true first integrals and configurational invariants. True first integrals must exist on M and are a subclass of the configurational invariants.

Hietarinta,[3] implicitly, draws still another distinction. In his criticism of the use of polynomial forms for the variational determination of configurational invariants, he would require that the invariant forms first be solved for the integral constants; but then of course they may no longer be polynomial in the momenta. The point is most easily explained by example, but I hold that the resulting constraint is overly confining and adds little that is helpful to the definition of a class of invariants. In any event, Hietarinta's invariants can be converted to polynomial form.

The three Hamiltonians considered by Hietarinta are

$$H^A = \tfrac{1}{2}p_x^2 + \tfrac{1}{2}(p_y - x/y)^2 - \tfrac{1}{2}x^2/y^2, \tag{2}$$

$$H^B = \tfrac{1}{2}p_x^2 + \tfrac{1}{2}p_y^2 + 2yp_xp_y - x, \tag{3}$$

$$H^C = \tfrac{1}{2}p_x^2 + \tfrac{1}{2}p_y^2 + x/y. \tag{4}$$

In the first of these cases, Hietarinta finds the invariant form

$$I_2^A = (xp_y - yp_x + y)/p_y. \tag{5}$$

But then

$$J^A = (x - I_2^A)p_y - yp_x + y = 0 \tag{6}$$

is also invariant; J^A is a polynomial of first degree in the momenta. Any rational invariant may be trivially converted to polynomial form.

A word of clarification is useful here. Let (p_0,q_0) denote the initial values of $(p_x,p_y,x,y) = (p,q)$. The trivial rearrangement

$$J^A = p_y[I_2^A(p,q) - I_2^A(p_0,q_0)]$$

displays the polynomial form of $J^A[p,q;I_2^A(p_0,q_0)]$

$=0$, an invariant relation. The equations of motion give the Poisson bracket

$$\{J^A, H^A\} = (xp_y/y^2)[I_2^A(p,q) - I_2^A(p_0,q_0)]$$

which vanishes because $I_2^A(p,q)$ is constant. However, this is not how we would normally calculate J^A. Once we have established the polynomial form of the invariant, other methods[1,7] can be used to actually find it. Similar arguments hold for the other examples, and in general.

The second Hamiltonian, though not of the form of Eq. (1) and therefore a little obscure physically, is more interesting. Hietarinta provides two invariant forms

$$I_2^B = p_y \exp(p_x^2),$$

$$I_3^B = -y\exp(-p_x^2) + \tfrac{1}{4}(2\pi)^{1/2} p_y \times \exp(p_x^2)\,\mathrm{erf}(2^{1/2}p_x).$$

Hence

$$y = [\tfrac{1}{4}(2\pi)^{1/2} I_2^B \,\mathrm{erf}(2^{1/2}p_x) - I_3^B]\exp(p_x^2) = F(p_x).$$

Invert this result. Then $p_x = f_j(y)$ where the subscript j counts all branches of the possibly-multiply-sheeted inversion $F^{-1}(y) = f_j(y)$. Thus we can construct

$$J^B = \prod[\tfrac{1}{2}p_y^2 + 2yf_j(y)p_y + \tfrac{1}{2}f_j^2(y) - x - E] = 0, \tag{7}$$

the product being taken over all branches of the inversion. The polynomial J^B is invariant.

Finally, in the last case, Hietarinta provides the invariant form

$$I_3^C = (p_y W_- + 2W_-')/(p_y W_+ + 2W_+')$$

where $W_\pm = W_\pm(\tfrac{1}{2}E, p_x)$ are the standard solutions[8] of the parabolic cylinder equation $\zeta''(p_x) + (\tfrac{1}{4}p_x^2 - \tfrac{1}{2}E)\zeta(p_x) = 0$. Here primes denote differentiation with respect to p_x and $H^C = E$. Thus, if $W(\tfrac{1}{2}E, p_x) = W_-(\tfrac{1}{2}E, p_x) - I_3^C W_+(\tfrac{1}{2}E, p_x)$, then

$$E - x/y = \tfrac{1}{2}p_x^2 + \tfrac{1}{2}p_y^2 = \tfrac{1}{2}p_x^2 + 2(W'/W)^2 = F(p_x).$$

Inverting as before, $F^{-1}(\zeta) = f_j(\zeta)$,

$$J^C = \prod[p_x - f_j(E - x/y)] = 0. \tag{8}$$

Once again, the polynomial J^C is invariant.

The procedures by which we derived the polynomial invariants for Hietarinta's Hamiltonians are easily generalized. The extension to integrable systems of arbitrary dimension is self-evident.[7]

If the multiplicity of branches is small, as is usual for useful invariants, J will be a polynomial of low degree. (Some skill may be required to avoid inversion combinations that contribute to a needlessly high multiplicity.) On the other hand, it is conceivable that sometimes a highly-multiply-sheeted expression is necessary. In that event, the actual orbital behavior could *appear* to be stochastic.[1,7]

It would be interesting indeed to discover a system having a known invariant and possessing a high intrinsic (nondegenerate) branching multiplicity. To this end, Hietarinta's third system almost fills the bill. Notice that $W(\tfrac{1}{2}E, p_x)$ is oscillatory when $E - \tfrac{1}{2}p_x^2$ is negative, nonoscillatory otherwise. Thus, a high multiplicity could occur for large negative energy. But the motion itself is nonoscillatory; the sheets are connected at infinity. Nearby orbits might diverge widely, but an individual trajectory would not appear to be stochastic.

In conclusion, the above results show the ubiquity of invariant forms that are polynomial in momenta for integrable Hamiltonians. Thus the search for invariants based upon procedures assuming the existence of such polynomial invariants is quite broadly applicable.

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[2]Ref. 1, the quadrature approach, Eq. (5.1), and to be published.

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[4]E. T. Whittaker, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies* (Cambridge Univ. Press, London, 1937), 4th ed., Chaps. 3 and 12.

[5]R. Abraham and J. E. Marsden, *Foundations of Mechanics* (Benjamin/Cummings, London, 1978), 2nd ed., Chap. 3.

[6]W. Sarlet, P. G. L. Leach, and F. Cantrijn, "Exact Versus Configurational Invariants and a Weak Form of Complete Integrability," to be published.

[7]L. S. Hall, in "Local and Global Methods of Dynamics," edited by R. Cawley, A. W. Saenz, and W. W. Zachary, Lecture Notes in Physics (Springer-Verlag, New York, to be published).

[8]*Handbook of Mathematical Functions,* edited by M. Abramowitz and I. Stegun (Dover, New York, 1972), Chap. 19.