Comments on "General Theory for Quantum Statistics in Two Dimensions"

In a recent Letter[1] Wu provides, among other results, a derivation of exotic quantum statistics in two-dimensional space using Feynman path integrals, extending an argument given by Laidlaw and De Witt[2] for the three-dimensional case. The existence of such statistics and their physical interpretation in terms of local observables was previously obtained from the standpoint of group representations.[3] However, from this standpoint more general quantum theories than those in Ref. 1 are recognized to occur. Within a single framework, one obtains in addition the theories of particles obeying parastatistics, as well as particles with spin.[4,5] The description of these theories requires extension of the conventional path-integral formalism.

It has been shown[6] that quantum mechanics in R^s can be described by unitary representations of $\mathrm{Diff}(R^s)$, the group of diffeomorphisms of R^s which become trivial at infinity. Quantum statistics arises from certain *induced* representations of this group. $\mathrm{Diff}(R^s)$ acts in a natural way on n-particle configuration space Δ, whose fundamental group $\pi_1(\Delta)$ is the braid group B_n for $s=2$ and the symmetric group S_n for $s>2$. Then $\pi_1(\Delta)$ serves as a gauge group for the theory, and its unitary representations induce representations of $\mathrm{Diff}(R^s)$ describing the various particles statistics.

Wu states that "all possible quantum statistics in two-space are characterized by an angle parameter θ which interpolates between bosons and fermions." This assertion presupposes representations of the braid group which are one dimensional. There are, however, quantum theories obtained as representations of $\mathrm{Diff}(R^s)$ induced by higher-dimensional representations of $\pi_1(\Delta)$, corresponding to parastatistics (for S_n) or "unusual parastatistics" (for B_n). It does not seem widely recognized[7] that parastatistics[8] can also be described by Feynman path integrals on configuration space, taking the wave function ψ to be vector valued rather than scalar valued, and the propagator K to be an operator-valued function. Then the "weights" $\chi(\alpha)$ in Ref. 1, Eq. (1), for $\alpha\in\mathrm{B}_n$ or $\alpha\in\mathrm{S}_n$, can be unitary operators instead of phases, while $\int\exp(iS)Dq$ remains a scalar quantity.

It should also be noted that even systems of *distinguishable* particles can be described by quantum theories with unusual phase shifts in two-dimensional space, because the *coordinate* space $\{(\mathbf{x}_1,\ldots,\mathbf{x}_N)\,|\,\mathbf{x}_i\in R^2, \mathbf{x}_i\neq\mathbf{x}_j \text{ for } i\neq j\}$ is not simply connected.[9] A possible example is that of quantized disturbances such as vortices in a thin film. Now *different* phase shifts can occur when different pairs of vortices circle each other by means of continuous paths in coordinate space. These phase shifts may be related to the relative vorticities. Thus it is not really the indistinguishability of the particles which accounts for the occurrence of unusual statistics in R^2, but the two-dimensionality of the space.

Quantum theories of particles with spin are also obtained as induced representations of $\mathrm{Diff}(R^2)$ or $\mathrm{Diff}(R^3)$. In one-particle configuration space, the gauge groups are the universal covering groups of the Lie groups $\mathrm{SL}(2,R)$ or $\mathrm{SL}(3,R)$ respectively.[5] Unitary representations of these groups can be decomposed with respect to the covering groups of SO(2) or SO(3). Now it is essential to consider the higher-dimensional representations in carrying out the inducing construction. In three dimensions, we obtain quantum theories of supermultiplets of particles with integer or half-integer spin, and in two dimensions particles with fractional spin, strictly from the representation theory of the diffeomorphism group. To express these systems in terms of path integrals, it appears necessary to enlarge the configuration space.[10]

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[6]The theoretical foundations for considering the group of diffeomorphisms of R^s in quantum mechanics are summarized in G. A. Goldin, R. Menikoff, and D. H. Sharp, Phys. Rev. Lett. **51**, 2246 (1983), and are also discussed in the papers in Ref. 5.

[7]See Ref. 2, as well as L. S. Schulman, *Techniques and Applications of Path Integration* (Wiley, New York, 1981).

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[9]There is a natural homomorphism $h\colon \mathrm{B}_n\to\mathrm{S}_n$. The subgroup of B_n which maps to the identity in S_n is the fundamental group of this coordinate space.

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