Collective-Excitation Gap in the Fractional Quantum Hall Effect

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(Received 25 October 1984)

We present a theory of the collective excitation spectrum in the fractional quantum Hall-effect regimes, in analogy with Feynman's theory for helium. The spectrum is in excellent quantitative agreement with the numerical results of Haldane. *Within this approximation* we prove that a finite gap is generic to any liquid state in the extreme quantum limit and that in this single-mode *approximation* gapless excitations can arise only as Goldstone modes for ground states with broken translation symmetry.

PACS numbers: 72.20.My, 71.45.Gm, 73.40.Lq

The fractional quantum Hall effect[1] (FQHE) is one of the most remarkable many-body phenomena discovered in recent years. Associated with the quantization of the Hall resistance is a nearly complete freedom from dissipation. The latter suggests the existence of an excitation gap, presumably due to many-body correlations arising from the Coulomb interaction. Considerable theoretical effort has been made to understand the nature of the ground state which, at least for values of the Landau-level filling factor of the form $\nu = 1/m$, where m is an odd integer, seems to be quite well described by Laughlin's variational wave function.[2] In this Letter we present a theory of the excitation spectrum in the FQHE analogous to Feynman's theory for the excitation spectrum of superfluid ^{4}He.[3]

The Feynman argument for the excitation energy is equivalent to the assumption that the dynamic structure factor[4]

$$S(k,\omega) = \sum_n |\langle n|\rho_k|0\rangle|^2 \delta(E_n - E_0 - \omega) \tag{1}$$

is of the form

$$S(k,\omega) = Ns(k)\delta(\omega - \Delta(k)) \tag{2}$$

where $\rho_k \equiv \sum_j \exp(i\mathbf{k}\cdot\mathbf{r}_j)$. In this single-mode approximation (SMA) the excitation energy is

$$\Delta(k) = f(k)/s(k), \tag{3}$$

where the oscillator strength is

$$f(k) = N^{-1}\int d\omega\, \omega S(k,\omega), \tag{4}$$

and $s(k)$ is the static structure factor.

We may gain some insight into the validity of the SMA by noting that it works well in a variety of systems. In superfluid ^{4}He it is exact at long wavelengths and gives a good approximation to the entire phonon-roton excitation curve.[3] For the three- and two-dimensional electron gas (no magnetic field) it is again an excellent approximation to the plasmon at long wavelengths and a rough fit to the entire single-particle plasmon continuum at shorter wavelengths. For the two-dimensional electron gas in a large magnetic field it gives an accurate description at long wavelengths of the magnetoplasmon mode near $\omega_c \equiv eH/mc$. In general the SMA is accurate at long wavelengths where the oscillator strength in continuum modes is small or wherever these continuum modes do not exist.

For the FQHE high-energy cyclotron modes are not of primary interest. Of more relevance to the experiment and the nature of ground-state correlation are the low-lying excitations. Equation (3) tells us very little about such modes. However, if we insist that the excited states $|n\rangle$ in Eq. (1) lie within the lowest Landau level we get a version of Eq. (3) (the projected SMA), which describes these low-lying excitations. To do this we replace ρ_k by its projection $\bar{\rho}_k$ (bars indicate projected quantities), i.e.,

$$\Delta(k) = \bar{f}(k)/\bar{s}(k). \tag{5}$$

With use of the projected density operator[5] ($z_j = x_j + iy_j$, $k = k_x + ik_y$)

$$\bar{\rho}_k = \sum_{j=1}^{N} \exp(ik\partial/\partial z_j)\exp(ik^* z_j/2), \tag{6}$$

$\bar{s}(k)$ is easily shown to be

$$\bar{s}(k) = s(k) - (1 - e^{-|k|^2/2}) \tag{7}$$

$[\hbar = l \equiv (ec/H)^{1/2} = 1]$.

To find the projected oscillator strength we manipulate Eqs. (1) and (4) in the standard way,[4] i.e.,

$$\bar{f}(k) = N^{-1}\langle 0|\bar{\rho}_k^\dagger[\bar{H},\bar{\rho}_k]|0\rangle, \quad (8)$$

where the projected Hamiltonian is

$$\bar{H} = \tfrac{1}{2}\int [d^2q/(2\pi)^2] v(q) [\bar{\rho}_q^\dagger \bar{\rho}_1 - \rho e^{-q^2/2}]. \quad (9)$$

Under the assumption that the electrons are embedded in a solid with a static dielectric ϵ_0, $v(q) = 2\pi e^2/(\epsilon_0 q)$. The commutation in Eq. (8) may be computed to yield

$$\bar{f}(k) = (\nu/2\pi)\int [d^2q/(2\pi)^2] v(q) \int d^2r [g(r)-1][e^{-|k|^2/2} e^{i\mathbf{q}\cdot\mathbf{r}}(e^{(kq^*-k^*q)/2}-1) + e^{i(\mathbf{k}+\mathbf{q})\cdot\mathbf{r}}(e^{\mathbf{k}\cdot\mathbf{q}} - e^{k^*q})], \quad (10)$$

where $g(r)$ is the two-point correlation function related to $s(k)$ (for a homogeneous and isotropic system) by

$$s(k) = 1 + \rho\int d^2r e^{i\mathbf{k}\cdot\mathbf{r}}[g(r)-1] + \rho(2\pi)^2\delta^2(k). \quad (11)$$

We have thus succeeded in expressing $\Delta(k)$, the excitation energy, in terms of quantities dependent solely on the ground state. Since the kinetic energy has been quenched by the magnetic field the scale of energy is set solely by the scale of the interaction $[e^2/(\epsilon_0 l)]$.

Expansion of Eq. (10) shows that $\bar{f}(k)$ vanishes as $|k|^4$. Examination of (7) and (11) shows that $\bar{s}(k)$ also vanishes as $|k|^4$ if $M_0 = M_1 = -1$, where

$$M_n = \rho\int d^2r (r^2/2)^n [g(r)-1]. \quad (12)$$

In the symmetric gauge, at any filling factor ν, $g(r)$ may be written as

$$\rho[g(r)-1] = (2\pi\nu)^{-1}\sum_{s=0}^{\infty}(r^2/2)^s \exp(-r^2/2)/s![\langle n_s n_0\rangle - \langle n_s\rangle\langle n_0\rangle - \nu\delta_{s0}], \quad (13)$$

where n_s is the occupation number for the sth angular momentum state. Substitution of (13) into (12) yields

$$M_0 = \nu^{-1}[\langle N n_0\rangle - \langle N\rangle\langle n_0\rangle] - 1, \quad (14)$$

$$M_1 = \nu^{-1}[\langle (L+N) n_0\rangle - \langle L+N\rangle\langle n_0\rangle] - 1, \quad (15)$$

where $N = \sum_{s=0}^{\infty} n_s$ and $L = \sum_{s=0}^{\infty} s n_s$ are the total particle number and angular momentum. Since L and N are constants of the motion, their fluctuations vanish leaving $M_0 = M_1 = -1$. This general result implies that for any homogeneous and isotropic ground state $\bar{s}(k) \approx |k^4|$ for $k \to 0$. In order to relate this ground-state property to the excitation spectrum we must use the SMA which may only be approximate. However, it seems *plausible* that in this system there can be no low-lying single-particle excitations to defeat this gap. The kinetic energy necessary to produce such excitations has been quenched by the magnetic field. Hence we can conjecture that the only way that gapless excitations can occur is as Goldstone modes in systems with broken translation symmetry (e.g., the Wigner crystal[6,7]). In this approximation, it would appear that the existence of a gap for liquid ground states is the *rule* rather than the exception. Whether or not liquid ground states must have rational ν is an entirely separate question, about which nothing has been proved.

In order to evaluate Eq. (5) using (10) and (11) we need a specific model for the ground state. We have chosen to use the Laughlin ground state (LGS) for $\nu = \frac{1}{3}, \frac{1}{5}$.[2] For the LGS $\bar{s}(k)$ does vanish as $|k|^4$ with a coefficient which may be calculated exactly, i.e.,[8,9]

$$\bar{s}(k) = |k|^4(1-\nu)/8\nu. \quad (16)$$

For the LGS we chose to fit an analytic parametrization[10] for $g(r)$ to the Monte Carlo simulation data of Ref. 7. The resulting gap functions from Eq. (5) for $\nu = \frac{1}{3}, \frac{1}{5}$ are plotted in Fig. 1. The deep minimum in the gap dispersion is caused by a peak in $\bar{s}(k)$ and is, in this sense, quite analogous to the roton minimum in helium.[3] We interpret the deepening of the minimum in going from $\nu = \frac{1}{3}$ to $\nu = \frac{1}{5}$ to be a precursor of the collapse of the excitation gap which occurs at the critical density for Wigner crystallization[6] ($\nu \sim \frac{1}{6.5}$). Further evidence for this interpretation is provided by the fact that the magnitude of the primitive reciprocal lattice vector for the crystal lies close to the roton minimum, as indicated by the arrows in the figure.

Note that the $\nu = \frac{1}{3}$ results are in excellent agreement with the small system ($N = 6, 7$ particles) numerical calculations of Haldane.[11] Note also that in contrast to the case of helium the SMA works well without explicit backflow corrections even in the region of the roton minimum.[3] This can be understood from a semiclassical point of view (which can be shown to be valid for the lowest Landau level) in which the local current density has the form $\mathbf{J}(\mathbf{r}) = \rho(\mathbf{r})\nabla\phi(\mathbf{r})\times\hat{\mathbf{B}}$, where ϕ is the local potential. The current density around a particular charge then satisfies $\nabla\cdot\mathbf{J} = 0$ so that the backflow condition[3] (con-

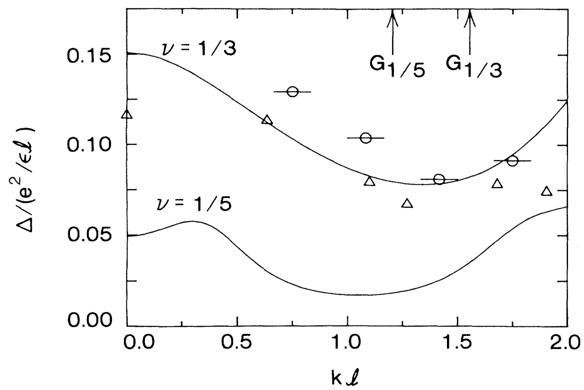

FIG. 1. Gap Δ vs wave vector for $\nu=\frac{1}{3},\frac{1}{5}$. Circles are from $N=7$ small spherical system calculations of Ref. 10. Horizontal error bars indicate uncertainty in conversion of angular momentum on a sphere to linear momentum. Triangles are for $N=6$ periodic boundary condition calculations with a hexagonal unit cell of Ref. 10. The arrows indicate the magnitude of the primitive reciprocal-lattice vectors of the hexagonal Wigner crystal, for $\nu=\frac{1}{3},\frac{1}{5}$. No small system calculations exist for $\nu=\frac{1}{5}$.

tinuity equation) is automatically satisfied.

For wave vectors beyond the roton minimum the SMA rapidly breaks down and it can be shown that the exact first moment of the excitation spectrum saturates at a large finite value $2|E_c|/(1-\nu)$, where E_c is the cohesive energy. Nevertheless, it is possible for us to estimate the excitation gap at $k=\infty$ by supposing that the lowest collective mode crosses over at the roton minimum from being a pure density oscillation to a bound quasiparticle-quasihole exciton.[9] The asymptotic exciton dispersion is[9] $\Delta_\mu(k)=\Delta_\mu-\nu^3/k$. Equating this to the SMA approximation to the gap at the minimum yields $\Delta_{1/3,1/5}=0.106, 0.025$. These values lie considerably above the results of hypernetted chain calculations of Laughlin,[2] $\Delta_{1/3,1/5}=0.057, 0.014$, and of Chakraborty,[12] $\Delta_{1/3,1/5}=0.053, 0.014$. However, preliminary Monte Carlo results of Morf and Halperin[13] yield a larger value, $\Delta_{1/3}=0.094\pm0.005$. Haldane's small system calculations[11] yield a value (extrapolated to $N=\infty$) of $\Delta_{1/3}=0.105\pm0.005$, in excellent agreement with the present result. Meaningful comparison of these results with experimental activation energies[14] must await a deeper understanding of the role of disorder.

The authors are grateful to F. D. M. Haldane for supplying them with his numerical results prior to publication. One of us (S.M.G.) would like to acknowledge several conversations with R. B. Laughlin concerning the implications of the present results and would also like to thank D. K. Kahaner, S. Leigh, and J. Filliben for assistance with the numerical analysis and graphics. This work was begun at the Aspen Center for Physics during the 1984 workshop on the quantum Hall effect.

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