Quantum Electrodynamic Energy Shifts of Quarks Bound in a Cavity

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The lowest-order quantum electrodynamic corrections to the energy levels of quarks bound in a spherical cavity are evaluated over a wide range of values of the quark mass. It is found that the self-energy is positive for massless quarks and decreases smoothly as the quark mass increases. The interaction energy between quarks is also calculated and found to dominate the self-energy correction. For this model, the vacuum polarization correction vanishes.

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The problem of understanding the electromagnetic mass shifts of hadrons is a very old one. The fact that the proton mass differs from the neutron mass by $0.59(\alpha/\pi)m_p$ suggests that the small mass difference might be calculable as an electromagnetic effect, although the sign of the effect is opposite to what one would expect.[1] The essential difficulty of the problem is that although the behavior of the photon, whose emission and reabsorbtion gives rise to the lowest-order electromagnetic mass shift, is well understood, the behavior of the hadronic system in between these interactions depends on the far less well understood strong interactions. In earlier approaches this problem of accounting for hadronic structure was recast by the use of the Cottingham formula[2] into the problem of understanding inelastic photon-hadron scattering, which is partially accessible experimentally. However, given the weight of evidence in favor of the quark model, the problem in modern terms can be thought of as one of determining the electromagnetic corrections of these charged constituents, since the strong forces binding quarks appear to be electromagnetically neutral. In this paper, we present a calculation of the lowest-order quantum electrodynamic corrections to quarks bound in hadrons. In order to deal with the strong interaction dynamics of the quarks, we work in a model where the confinement of the quarks in a nucleon is provided by an infinitely deep spherical scalar potential square well. Inside the well, the quarks are noninteracting in the absence of electromagnetism. As regards the quarks, this approximation is equivalent to the cavity approximation to the MIT bag model.[3] Within this model, the formalism is analogous to that of the problem of a few electrons bound in a strong Coulomb field.[4] In particular, to lowest order in α, the corrections are given by the Feynman diagrams in Fig. 1. In these diagrams, the double lines represent the wave function and propagation function of the quark in the confining potential. The propagation function, which has been studied extensively recently,[5,6] can be constructed with well known methods that have been applied to the calculation of quantum electrodynamic corrections to electrons in a strong Coulomb field.[7-10] Recent works have also examined quantum chromodynamic corrections with similar methods.[11-14] In the remainder of this paper, we outline the calculation of the quantum electrodynamic corrections in Fig. 1, and give the results. A detailed account of this work will be given elsewhere.

In the model considered here, the quark wave functions are solutions of the Dirac equation

$$(-i\boldsymbol{\alpha}\cdot\nabla+\beta m+V)\psi_n(\mathbf{r})=E_n\psi_n(\mathbf{r}), \tag{1}$$

where

$$V=\begin{cases}0, & r<R,\\ \beta V_0, & r>R,\end{cases} \tag{2}$$

in the limit where $V_0\rightarrow\infty$. In the limit, this potential is equivalent to the bag-model boundary condition

$$(i\boldsymbol{\gamma}\cdot\hat{\mathbf{r}}+1)\psi_n(\mathbf{r})=0 \tag{3}$$

at $|\mathbf{r}|=R$, and the limiting process provides a means of resolving ambiguities near the cavity surface. The spectrum consists of discrete eigenvalues in the intervals $(-\infty,-m)$ and (m,∞). The eigenvalues for angular momentum quantum number κ are the values of

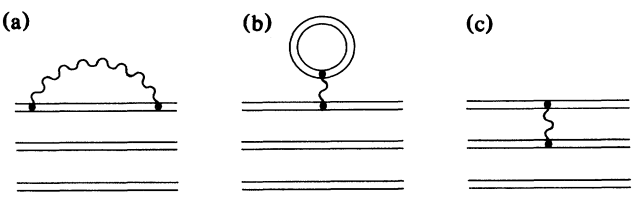

FIG. 1. Feynman diagrams contributing to the electromagnetic energy of a nucleon.

z for which

$$(z-m)^{1/2}j_{\kappa_+}(cR)=(m+z)^{1/2}j_{\kappa_-}(cR),\quad c=(z^2-m^2)^{1/2},\quad \kappa_+=|\kappa+\tfrac{1}{2}|-\tfrac{1}{2},\quad \kappa_-=|\kappa-\tfrac{1}{2}|-\tfrac{1}{2} \tag{4}$$

is satisfied.

The self-energy correction represented by the Feynman diagram in Fig. 1(a), which is the main part of the calculation, is given in coordinate space for a quark of charge ϵe by

$$\Delta E_{\rm SE}=-i(\epsilon^2\alpha/2\pi)\int d\mathbf{x}_2\int d\mathbf{x}_1\psi_n^\dagger(\mathbf{x}_2)\alpha_\mu\int_C dz\, G(\mathbf{x}_2,\mathbf{x}_1,z)\alpha^\mu\psi_n(\mathbf{x}_1)(e^{-bx_{21}}-e^{-b'x_{21}})/x_{21}$$
$$-\delta m\int d\mathbf{x}\,\psi_n^\dagger(\mathbf{x})\beta\psi_n(\mathbf{x}) \tag{5}$$

in the limit $\Lambda\to\infty$ for the Pauli-Villars regulator parameter. In (5), G is the Green's function associated with Eqs. (1)–(3), $b=-i[(E_n-z)^2+i\epsilon]^{1/2}$, $\mathrm{Re}(b)>0$, $b'=-i[(E_n-z)^2-\Lambda^2+i\epsilon]^{1/2}$, $\mathrm{Re}(b')>0$, $\delta m=m(\alpha/\pi)[\frac{3}{4}\ln(\Lambda^2/m^2)+\frac{3}{8}]$, and $x_{21}=|\mathbf{x}_2-\mathbf{x}_1|$. The contour of integration over z is shown in Fig. 2. In the case of massless quarks, for which no mass renormalization is needed, this correction was considered by Chodos and Thorn,[15] and more recently by Goldhaber, Hansson, and Jaffe.[11] For massive quarks there is an infinite mass renormalization which is carried out here with the standard method employed in calculations of the Lamb shift. The quark Green's function has the important feature of separating naturally into a free Green's function and a boundary-dependent correction. This separation allows for a straightforward treatment of the infinite mass renormalization, since the infinite part arises entirely from the free term. It should be noted that in a self-energy calculation with an arbitrary external potential, infinite wave-function and vertex renormalization terms arise and must be explicitly canceled against each other. However, such terms are proportional to $\langle V\rangle$ which happens to vanish in this model. The calculation thus divides into two parts. In the first part, the contribution to the self-energy from the free Green's function is calculated with the aid of the explicit expression of this function. This term was evaluated in three different ways. The most accurate method gave an estimated accuracy of 0.05%, and agreed with the other two methods within their error bars. The results are listed in Table I, second column.

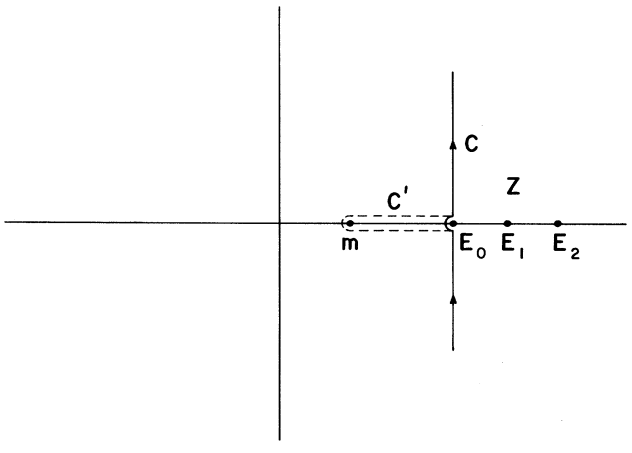

FIG. 2. The contour of integration in the complex z plane. The dotted line that circles the branch point of the free Green's function is not included in this calculation.

To evaluate the boundary-dependent term, the corresponding part of the Green's function is written as a sum over eigenfunctions of angular momentum. The integration over coordinate angles in (5) is carried out leaving a three-dimensional integral over the two radial coordinates and z. The integration over z is indicated by the contour in Fig. 2; care must be taken with the infinitesimal semicircle at the origin, because the ground-state pole gives a nonvanishing contribution for this segment of the contour. The calculation over the rest of the contour was done with two methods based on independent code. In the more accurate method, when the angular momentum summation is cut off, an analytic estimate of the remainder based on the asymptotic form of the terms in the sum is added. This summation requires up to several thousand terms to achieve an accuracy of 1 part in 10^4 in the integrand in the critical region where $x_1\approx x_2\approx R$. The integrations are then carried out numerically by Gaussian quadrature.

TABLE I. Self-energy level shift in units of $\epsilon^2\alpha/\pi R$.

mR	Free	Bound	Sum
0	0.618	2.452	3.070
0.1	0.381	2.440	2.821
0.5	−0.130	2.427	2.297
1	−0.602	2.466	1.864
1.5	−1.002	2.538	1.536
2	−1.347	2.625	1.279
3	−1.892	2.800	0.908
4	−2.284	2.950	0.666
6	−2.775	3.167	0.392
8	−3.051	3.304	0.253
10	−3.221	3.397	0.176

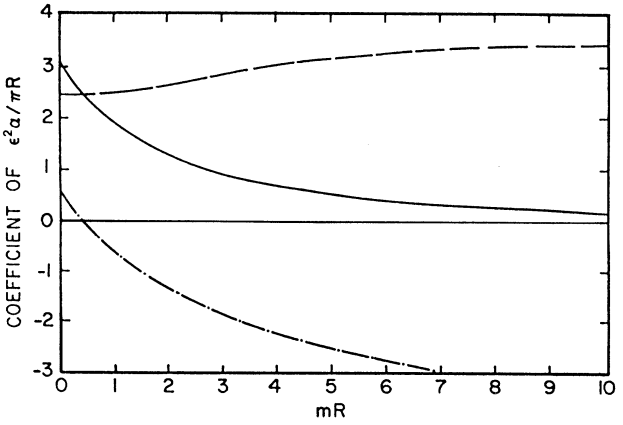

FIG. 3. The self-energy for quarks of charge ϵe. The solid line is the total level shift in units of $\epsilon^2\alpha/\pi R$, the dash-dotted line is the free-propagator contribution, and the dashed line is the remainder.

The results, shown in the third column of Table I, have an estimated uncertainty of 1 or less in the third place past the decimal. The free and bound parts of the self-energy and the sum are plotted in Fig. 3 for the range of quark masses $0 < mR < 10$. We note that the self-energy is positive for small masses and decreases smoothly toward zero for large masses. We obtain a smooth behavior near $m = 0$ with a limiting value of $\Delta E_{\rm SE} = (\epsilon^2\alpha/\pi R)3.070$ at $m = 0$. This result is consistent with the original calculation of Chodos and Thorn[15] who obtained $\Delta E_{\rm SE} = (\epsilon^2\alpha/\pi R)2.4(3)$, and with recent results of Goldhaber, Hansson, and Jaffe.[16] Current results by Baacke and Usler[17] for $m = 0$ and $m \neq 0$ based on methods described in Ref. 12 are in agreement with the results in Table I, third column.

The vacuum polarization level shift represented by the Feynmann diagram in Fig. 1(b), for a quark of charge ϵe in the loop, can be obtained from the vacuum polarization charge density[18]

$$\rho_{\rm VP}(x) = -\tfrac{1}{2}\epsilon e\,[\textstyle\sum_{n_+}|\psi_n(x)|^2 - \sum_{n_-}|\psi_n(x)|^2], \qquad (6)$$

where ψ_n are the solutions of Eqs. (1)–(3), n_+ denotes summation over positive-energy eigenfunctions, and n_- denotes summation over negative-energy eigenfunctions. This charge density vanishes identically as shown by the following considerations. The eigenvalue condition (4) is invariant under the simultaneous replacements $z \to -z$ and $\kappa \to -\kappa$. Also, in the radial Dirac equation, the same replacements exchange upper and lower components, with the result that $|\psi_{E,\kappa}|^2 = |\psi_{-E,-\kappa}|^2$ and

$$\sum_{n_+,\kappa}|\psi_{E,\kappa}|^2 = \sum_{n_+,\kappa}|\psi_{-E,-\kappa}|^2 = \sum_{n_-,\kappa}|\psi_{E,\kappa}|^2, \qquad (7)$$

and so $\rho_{\rm VP}(x) = 0$.

The electromagnetic interaction energy between two quarks of charge $\epsilon_1 e$ and $\epsilon_2 e$ is given by

$$\Delta E_{\rm EM} = \epsilon_1\epsilon_2\alpha\int(d\mathbf{x}_1 d\mathbf{x}_2/x_{21})\psi_1^\dagger(\mathbf{x}_1)\alpha_\mu\psi_1(\mathbf{x}_1)\psi_2^\dagger(\mathbf{x}_2)\alpha^\mu\psi_2(\mathbf{x}_2), \qquad (8)$$

corresponding to Fig. 1(c). This can be parametrized as

$$\Delta E_{\rm EM} = \epsilon_1\epsilon_2(\alpha/\pi R)[f_1(mR) + f_2(mR)\langle\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\rangle], \qquad (9)$$

where f_1 and f_2 are tabulated in Table II for quarks of equal mass in the ground state.

TABLE II. Calculated values for the functions f_1 and f_2.

mR	f_1	f_2
0	4.016	−0.428
0.1	4.045	−0.417
0.5	4.159	−0.375
1	4.299	−0.325
1.5	4.430	−0.279
2	4.549	−0.239
3	4.748	−0.175
4	4.900	−0.130
6	5.100	−0.077
8	5.221	−0.050
10	5.298	−0.035

The results give the lowest-order electrodynamic corrections to bound quarks for the model considered. It is important to understand these corrections, as this extra energy must be taken into account with the proton-neutron mass difference to infer the mass splitting between the up and down quarks. If the up and down quarks had equal mass, then, since the self-energy is proportional to the square of the charge of the quark, a negative self-energy would be necessary to decrease the energy of the up quark with charge $2e/3$ relative to the down quark with charge $-e/3$ in order to explain the sign of the neutron-proton mass difference. The present results give a positive self-energy for all masses considered, and since the interaction energy also increases the proton mass relative to the neutron mass, the explanation of the mass difference in this model remains the standard one—that the down quark is heavier than the up quark.

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