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Curious New Statistical Prediction of Quantum Mechanics

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Whether quantum mechanical systems, within the interval between two measurements, can be assigned definite, simultaneous values of noncommuting observables has been the subject of a very old dispute. A certain widely held assumption about such systems has been crucial to that dispute. That assumption turns out to be wrong: A previously unknown prediction of quantum mechanics, which fails to satisfy that assumption, is described here, and the consequences of that failure are considered.

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For half a century now, it has been argued by some authors (and disputed by others)[1] that there is a profound and intriguing distinction in quantum mechanics between what can be known about the past and what can be predicted about the future. What is at issue is the description of quantum mechanical systems within the time interval *between* two measurement events. The reasoning runs essentially this way: Consider a quantum mechanical system whose Hamiltonian, for simplicity, we shall take to be zero. Suppose that this system is measured at time t_i to be in the state $|A=a\rangle$ (where A represents some complete set of commuting observables of the system, and a represents some particular set of eigenvalues of those observables), and is measured at time t_f ($t_f > t_i$) to be in the state $|B=b\rangle$. What do these results imply about the results of other experiments that might have been carried out within the interval ($t_i < t < t_f$) between them? It turns out that the probability (which was first written down by Aharonov, Bergmann, and Lebowitz[2]) that a measurement of some complete set of observables C within that interval, *if* it were carried out, would find that $C=c_j$ is

$$P(c_j) = \frac{|\langle A=a|C=c_j\rangle|^2 |\langle C=c_j|B=b\rangle|^2}{\sum_i |\langle A=a|C=c_i\rangle|^2 |\langle C=c_i|B=b\rangle|^2}; \qquad (1)$$

and that formula entails, among other things, that $P(a)=P(b)=1$. Consequently, these authors maintain that such a system, within such an interval, must have definite, dipersion-free values of *both* A and B, whether *or not* A and B may happen to commute. In their view, the proper quantum mechanical descriptions of the past and the future are essentially different: Our knowledge of the past is not restricted, in the same way as our ability to predict the future, by the uncertainty relations; indeed, so far as the past is concerned, the quantal formalism itself *requires* that those relations be violated.

On the other hand, the detractors of this view point to famous arguments in the literature of hidden variable theory (arguments due to Gleason,[3] Kochen and Specker,[4] and others) which seem to establish that certain sets of noncommuting observables can *never* simultaneously be well defined. These arguments involve an innocuous-looking assumption about the results of measurements of certain projection operators (of which more will be said later). Bell,[5] some years ago, remarked that hidden variable theories could in principle be imagined which (by some averaging over the values of those variables) reproduce the statistical predictions of quantum mechanics, and which nonetheless (when some *particular* values of those variables are assumed) fail to satisfy that assumption; but the quantum mechanical statistics *themselves, without* any other such addenda, have always been thought to satisfy it, and must therefore (so the argument goes) satisfy whatever that assumption entails: There must necessarily be a contradiction, according to these detractors, in the supposition that in a system such as was described above (a system which, by *assumption,* is purely quantum mechanical), $A = a$ and $B = b$ within the interval $t_i < t < t_f$.[6]

Is it somehow mistaken, then, or somehow misleading, to suppose that (1) attributes definite values to A and B? Is it that (1) itself produces some contradiction? How? Where?

No. It turns out (and this is the subject of the present note) that there is a remarkable and heretofore unknown property of the quantal statistics whereby quantum mechanical systems, within the interval between two measurements, *fail* to satisfy that assumption (the assumption about the projection operators), and so evade its consequences.

The assumption in question is this: Suppose that P_Ψ represents the projection operator $|\Psi\rangle\langle\Psi|$. The arguments of Gleason and of Kochen and Specker assume that if a measurement of P_Ψ on a certain system will with certainty yield $P_\psi = 0$, and if a measurement of P_ϕ on the same system (at the same time) will also, with certainty, yield $P_\phi = 0$, then a measurement (on that system at that time) of P_γ, where

$$|\gamma\rangle = a|\Psi\rangle + b|\phi\rangle, \tag{2}$$

will (for arbitrary choices of a and b), with certainty, yield $P_\gamma = 0$ as well.[7]

That seems reasonable enough. It is surely true, anyway, of the quantum mechanics of prediction of the future (any quantum state $|\delta\rangle$, after all, has the property that if $P_\psi|\delta\rangle = 0$ and $P_\phi|\delta\rangle = 0$, then $P_\gamma|\delta\rangle = 0$). But the quantum mechanics of retrodiction of the past is quite another matter.

Suppose that our system consists of a particle which may be located within any one of three small impenetrable boxes, placed, respectively, at x_1, x_2, and x_3 (the arguments of Gleason and of Kochen and Specker can only be formulated in Hilbert spaces of three or more dimensions, so that we shall have to consider such spaces in order to refute them). At t_i the particle is measured to be in the state

$$|A = a\rangle = \tfrac{1}{2}\sqrt{2}[|x_1\rangle + |x_2\rangle], \tag{3}$$

and at t_f the particle is measured to be in the state

$$|B = b\rangle = \tfrac{1}{2}\sqrt{2}[|x_2\rangle + |x_3\rangle]. \tag{4}$$

At intermediate times, according to (1), it will with certainty be the case that $A = a$ and $B = b$. Furthermore, since $\langle A = a|x_3\rangle = \langle B = b|x_1\rangle = 0$, (1) entails that at such times, with certainty, $X = x_2$. (*Three* noncommuting observables, then, can be simultaneously well defined for such a system within such an interval; and a little reflection will show that as the dimensionality of the Hilbert space is increased, the number of such observables will increase too, without any limit!)

Now something very curious arises. Consider an observable Q of our system, whose eigenstates are

$$\begin{aligned} |Q = q_1\rangle &= \tfrac{1}{2}\sqrt{2}[|x_1\rangle + |x_3\rangle], \\ |Q = q_2\rangle &= |x_2\rangle, \\ |Q = q_3\rangle &= \tfrac{1}{2}\sqrt{2}[|x_1\rangle - |x_3\rangle]. \end{aligned} \tag{5}$$

Suppose that Q is measured within the interval $t_i < t < t_f$. It might be expected, since $X = x_2$ within that interval, and since $|Q = q_2\rangle = |X = x_2\rangle$, that such a measurement will find, with certainty, that $Q = q_2$. But that is not so: Albeit $\langle A = a|x_3\rangle = \langle B = b|x_1\rangle = 0$, yet

$$\begin{aligned} \langle A = a|Q = q_1\rangle &\neq 0, \\ \langle B = b|Q = q_1\rangle &\neq 0, \end{aligned} \tag{6}$$

for $i = 1, 2, 3$. Consequently, albeit $P_{x_1} = 0$ and $P_{x_3} = 0$ within that interval, $P_{q_1} \neq 0$ there. The assumption of Gleason and of Kochen and Specker, as innocent as it looks, is not satisfied by quantum mechanical systems within the interval between two measurements!

Bell has pointed out that in spite of the argument of Gleason and of Kochen and Specker, and without violating the statistical predictions of quantum mechanics, it can consistently be supposed (within certain hidden-variable theories) that noncommuting observables can simultaneously be well defined. The present considerations suggest something stronger: In spite of that argument, and given those statistical predictions [given, particularly, Eq. (1)], it is inconsistent to suppose anything else.

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[1]An early example is A. Einstein, R. C. Tolman, and B. Podolsky, Phys. Rev. **37**, 780 (1931).

[2]Y. Aharonov, P. G. Bergmann, and J. L. Lebowitz, Phys. Rev. **134**, B1410 (1964).

[3]A. M. Gleason, J. Math. Mech. **6**, 885 (1957).

[4]S. Kochen and E. P. Specker, J. Math. Mech. **17**, 59 (1967).

[5]J. S. Bell, Rev. Mod. Phys. **38**, 447 (1966).

[6]This is a fairly tricky point; it ought to be made quite carefully. The arguments of Gleason and Kochen and Specker *do not* purport to establish that any two noncommuting observables cannot simultaneously be well defined (they *do* purport to establish that certain finite sets of such observables cannot simultaneously be well defined, but the cardinality of those sets is larger than two). On the other hand, those arguments (given the assumption about projection operators) *do* preclude the assignment of certain *particular* values, simultaneously, to certain such pairs of observables (see, for example, Ref. 5); and yet such assignments *can* arise as a consequence of Eq. (1). Suppose, then, that A and B are two such observables, with a and b two such particular values.

[7]Kochen and Specker suppose that the assignment of a definite value to every observable corresponds to the imbedding of the partial algebra of quantum observables into a commutative algebra of real-valued functions on a "phase space" of hidden variables; and therefrom follows the assumption about projection operators.