Soliton Motion in the Case of a Nonzero Reflection Coefficient

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A method is given for finding the shifts in position of the solitons for the case of nonzero reflection coefficient. Expressions for boost generators in terms of scattering data play a prominent role in the analysis. Phase-shift formulas which show the effect of the radiation component on the soliton motion are deduced for the nonlinear Schrödinger equation, the Korteweg–de Vries equation, and the sine-Gordon equation.

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One of the most important characteristics of nonlinear evolution equations solvable by the inverse scattering method is that their longtime asymptotic solutions are composed of two classes of entities, namely, solitons and dispersive wave trains (radiation). However, the rigorous analysis of this property is a difficult mathematical task which is still far from completed,[1-3] the main difficulty being the description of asymptotic solutions containing both solitons and radiation.[3] A particularly important question is to determine how solitons are affected by the radiation component. This subject had already been considered by Zakharov[4] in 1971, but there was no decisive progress until 1982, when Ablowitz and Kodama[3] solved the problem for the case of the Korteweg–de Vries (KdV) equation. Nevertheless, the procedure used by these authors, based on the finite-perturbation method, does not seem easy to apply to other integrable models.

The purpose of this Letter is to provide a convenient method for determining the soliton phase shifts in the presence of radiation. The strategy is illustrated in detail for the case of the nonlinear Schrödinger (NS) equation. My analysis of this model is based on some general properties of integral equations of the Marchenko type and on the expression of the generator K of pure Galilean transformations in terms of scattering data.[5] At this point it must be noted that besides the NS equation, several other relevant integrable nonlinear equations are invariant under the Galilean or Poincaré groups. In a series of papers I have recently found the expression in terms of scattering-data variables for the "boost" generator K associated with the sine-Gordon (sG) equation,[6] the KdV equation,[7] and the massive Thirring model.[8] As a consequence, the analysis performed for the NS equation applies in a completely similar form to these other models.

The inverse scattering method for solving the NS equation[9]

$$i\psi_t = -\psi_{xx} - 2|\psi|^2\psi, \tag{1}$$

is based on the resolution of the following integral equations of the Marchenko type:

$$b_1^*(t,x,y) - \int_0^\infty \omega^*(t,x+y+z)b_2^*(t,x,z)dz = \omega^*(t,x+y), \quad b_2^*(t,x,y) + \int_0^\infty \omega(t,x+y+z)b_1(t,x,z)dz = 0, \tag{2}$$

where the kernel ω is determined from the set of scattering data

$$S(t) = \{k_j = \xi_j + i\eta_j,\ c_j(t) = c_j\exp(4ik_j^2 t),\ j=1,\ldots,N;\ r(t,k) = r(k)\exp(4ik^2 t),\ k\in R\}$$

in the form

$$\omega(t,x) = 2\sum_j c_j(t)e^{2ik_jx} + \pi^{-1}\int_{-\infty}^{\infty} r(t,k)e^{2ikx}dk. \tag{3}$$

The corresponding solution of (1) is given by $\psi(t,x) = -b_1(t,x,+0)$. As $t\to\pm\infty$, $\psi(t,x)$ evolves[2] into a superposition of N freely moving solitons with velocities $v_j = -4\xi_j$, and a radiation component which decays like $|t|^{-1/2}$. My aim is to determine the parameters $q_j^\pm$ which characterize the asymptotic trajectories $q_j^\pm(t) \sim q_j^\pm + v_j t$ for the solitons as $t\to\pm\infty$.

The NS equation is a Galilean-invariant Hamiltonian system and its corresponding generator of pure Galilean transformations can be written as[5]

$$K = -\tfrac{1}{2}\int_{-\infty}^{\infty} x|\psi|^2dx = -\sum_j \ln|c_j\partial_k a(k_j)| - (2\pi)^{-1}\int_{-\infty}^{\infty}\ln|a(k)|\partial_k \arg b(k)dk, \tag{4}$$

where

$$a(k)=\prod_j \frac{k-k_j}{k-k_j^*}\exp\left[\frac{i}{2\pi}\int_{-\infty}^{\infty}\frac{\rho(q)}{q-k-i0}dq\right],\quad \mathrm{Im}k\geqslant 0, \tag{5a}$$

$$\rho(k)=\ln(1+|r(k)|^2),\quad b(k)=r(k)a(k),\quad \mathrm{Im}k=0. \tag{5b}$$

From (4) and (5) the following alternative expression for K is formed at once:

$$K=-\sum_j\left\{\ln\frac{|c_j|}{2\eta_j}+\sum_{n\neq j}\ln\left|\frac{k_j-k_n}{k_j-k_n^*}\right|-\frac{\eta_j}{2\pi}\int_{-\infty}^{\infty}\frac{\rho(k)}{|k-k_j|^2}dk\right\}$$
$$+\frac{1}{4\pi}\int_{-\infty}^{\infty}\rho(k)\partial_k\left\{\arg r(k)+\frac{1}{2\pi}\mathrm{P}\int_{-\infty}^{\infty}\frac{\rho(q)}{q-k}dq\right\}dk, \tag{6}$$

where P signifies principle value. We observe that for a pure one-soliton solution, ψ_{sol}, the position of its center is[9] $q(t)=(2\eta)^{-1}\ln[1c(t)/2\eta]$, and therefore $K[\psi_{\mathrm{sol}}(t)]=-2\eta q(t)$.

As a consequence of the Galilean invariance of the NS equation we may define mass and momentum functionals which have the following forms in terms of scattering data[5]

$$M=\tfrac{1}{2}\int_{-\infty}^{\infty}|\psi|^2dx=2\sum_j\eta_j-\frac{1}{2\pi}\int_{-\infty}^{\infty}\rho(k)dk,\quad P=-i\int_{-\infty}^{\infty}\psi^*\psi_x dx=-8\sum_j\eta_j\xi_j+\frac{2}{\pi}\int_{-\infty}^{\infty}k\rho(k)dk.$$

According to these expressions the NS field appears as a Galilean system composed of N particle with masses $2\eta_j$ and velocities $v_j=-4\xi_j$ (solitons), and a continuous mass distribution with density $-\rho(k)/2\pi$ and velocity $v_g(k)=-4k$ (radiation). The meaning of this spectrum of velocities associated with the scattering data can be understood through the asymptotic analysis of the NS field as follows. Because of the form of Eqs. (2) it is clear that the value of the NS field ψ at a given point (t,x_0) depends only on the restriction of the kernel $\omega(t,x)$ to the interval $[x_0,+\infty)$. On the other hand, if the evolution law of the scattering data is inserted into Eq. (3) the modulus of the jth term in the sum propagates with velocity $v_j=-4\xi_j$, while the group velocity of the Fourier modes in the integral term coincides with $v_g(k)=-4k$. Hence, given arbitrary numbers x_0 and v_0, as $t\rightarrow\pm\infty$ the restriction of ω to intervals of the form $I^{\pm}(t)=[x_0+(v_0\pm\epsilon)t,+\infty)$ with $\epsilon>0$ arbitrary, depends only on those scattering data with velocity v such that $\pm(v-v_0)>0$. In fact, one may prove that the contribution[10] to ω due to the remaining scattering data has a L^2 norm on $I^{\pm}(t)$ which vanishes asymptotically as $t\rightarrow\pm\infty$. We are now ready to study the soliton motion. To this end let us consider the lth soliton and let us denote by $\psi_{\pm}$ the parts of the NS field propagating to the right of the soliton as $t\rightarrow\pm\infty$. With use of our analysis above, the relevant kernels for characterizing $\psi_{\pm}$ through Marchenko equations (2), are

$$\omega_{\pm}(t,x)=2\sum_{j\neq l}\theta(\pm(v_j-v_l))c_j(t)e^{2ik_jx}+\pi^{-1}\int_{-\infty}^{\infty}\theta(\pm[v_g(k)-v_l])r(t,k)e^{2ikx}dk, \tag{7}$$

where θ stands for the step function. That is to say, the sets of scattering data associated with $\psi_{\pm}$ are

$$S_{\pm}(t)=\{k_j,c_j(t),j\ \text{such that}\ \pm(v_j-v_l)\rangle 0;\ \theta(\pm[v_g(k)-v_l])r(t,k),\ k\in R\}. \tag{8}$$

On the other hand, the sets of scattering data $S_{\pm}(t)\bigcup\{k_l,c_l(t)\}$ will correspond to the parts $\psi'_{\pm}$ of the NS field moving as $t\rightarrow\pm\infty$ with velocity v such that $\pm(v-v_l)\geqslant 0$. In other words, $\psi'_{\pm}$ result from the addition of the lth soliton to $\psi_{\pm}$. Therefore, as a result of the dispersive character of the radiation component and the localized form of the soliton, it is clear that as $t\rightarrow\pm\infty$ the difference between the values of the functional K as $\psi'_{\pm}$ and $\psi_{\pm}$ must be equal to the value $-2\eta_e q_l^{\pm}(t)$ of K at the lth soliton. In this way, by using identity (6) we obtain

$$q_l^{\pm}=\frac{1}{2\eta_l}\ln\frac{|c_l|}{2\eta_l}+\frac{1}{\eta_l}\sum_{j\neq l}\theta(\pm(v_j-v_l))\ln\left|\frac{k_l-k_j}{k_l-k_j^*}\right|-\frac{1}{2\pi}\int_{-\infty}^{\infty}\theta(\pm[v_g(k)-v_l])\frac{\rho(k)}{|k-k_l|^2}dk. \tag{9}$$

Hence the phase shift of the lth soliton as it interacts both with the other solitons and the radiation component is given by

$$q_l^{+}-q_l^{-}=\frac{1}{\eta_l}\sum_{j\neq l}\mathrm{sgn}(v_j-v_2)\ln\left|\frac{k_l-k_j}{k_l-k_j^*}\right|-\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{sgn}[v_g(k)-v_l]\frac{\ln[1+|r(k)|^2]}{|k-k_l|^2}dk. \tag{10}$$

The integral term in (10) exhibits the effect of the radiation component on the soliton motion.

The same analysis may be applied to other integrable nonlinear models for which the expression of a boost generator in terms of scattering data is available. Thus, for the case of the KdV equation[7]

$$u_t = -u_{xxx} + 6uu_x, \tag{11}$$

the spectrum of velocities associated with the scattering data is $v_j = 4\eta_j^2$, $v_g(k) = -12k^2$; then the contribution of the radiation to the soliton shifts turns out to be

$$\Delta q_l^{\mathrm{rad}} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\ln[1-|r(k)|^2]}{k^2+\eta_l^2} dk, \tag{12}$$

which is in agreement with the result obtained by Ablowitz and Kodama.[3,11] Finally, for the sine-Gordon equation[6]

$$\phi_{tt} - \phi_{xx} + \sin\phi = 0, \tag{13}$$

one finds that the scattering data have a spectrum of rapidities given by $\beta_j = -\ln(4|\lambda_j|)$, $\beta_g(\lambda) = -\ln(4|\lambda|)$. The soliton shifts due to the presence of the radiation component adopt the form

$$\Delta q_l^{\mathrm{rad}} = [2\pi(1+e^{2\beta_l})]^{-1} \int_{-\infty}^{\infty} \mathrm{sgn}[\beta_l - \beta_g(\lambda)] \frac{\ln[1+|r(\lambda)|^2]}{|\lambda-\lambda_l|^2} d\lambda. \tag{14}$$

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