Nonlinear Landau Heating by Ion-Bernstein Waves in Magnetically Confined Fusion Plasmas

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It is shown that during ion-Bernstein–wave heating experiments, nonlinear ion Landau damping absorbs efficiently the incident Bernstein waves in present-day tokamak and tandem-mirror plasmas. Further, this nonlinear absorption will dominate absorption by minority (impurity) ions.

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Heating of ions by absorption of ion-Bernstein waves[1–3] at the low ion-cyclotron–harmonic frequencies has been advocated recently by Ono *et al.*[3] Efficient ion-Berstein–wave heating was observed in the Nagoya Japanese Institute of Plasma Physics (JIPP) II-U tokamak where H^+ majority and $^4He^{2+}$ minority ions were used.[4] Thus, near the center of the plasma, column absorption should occur when $f=3f_{c\mathrm{He}}$, while $2f=3f_{c\mathrm{H}}$. However, the heating results were found to be independent of the $^4He^{2+}$ minority ion concentration. Nevertheless, wave absorption by impurity ions (C^{6+},O^{8+}) may have occurred. Bernstein-wave–heating results were also obtained in the Princeton ACT-1 low-temperature toroidal device.[5]

In this Letter it is shown that in present-day experiments nonlinear ion-cyclotron (Landau) damping would absorb the wave energy at $2f=3f_{c\mathrm{H}}$, even in the absence of any minority ion species. Furthermore, in the presence of minority (impurity) ions, the nonlinear absorption is greatly enhanced. A further advantage of nonlinear absorption is that it would allow the use of ion-Bernstein waves to heat the plasma at $\omega/\omega_{ci}\simeq 1.5$ or 2.5 in the central cell of tandem-mirror devices where the magnetic field is radially uniform.

The mechanism proposed here is the self-interaction of a Bernstein wave when the conditions $2\omega \simeq m\omega_{ci}(0)$ are satisfied, where $m\geqslant 3$ corresponds to odd integers.[6] In particular, in the present paper the cases of $m=3,5$ are examined in some detail. The relevant selection rules are given by a generalization of wave-wave-particle scattering in a magnetized plasma[6,7]:

$$\omega-\omega''-(k_{\parallel}-k_{\parallel}'')v_{\parallel}=m\omega_{ci}, \tag{1}$$

where ω ($\mathbf{k}$) and ω'' ($\mathbf{k}''$) are the frequencies and wave vectors of the two Bernstein waves, m is an integer, and the beat wave $\omega'(\mathbf{k}')$ ("quasimode" or "virtual wave") resonates with particles (where $\omega'=\omega\pm\omega''$, $\mathbf{k}'=\mathbf{k}\pm\mathbf{k}''$). In the case of $\omega>\omega''$, $\omega>\omega'$, one has the case of decay instability which was observed experimentally for electron Bernstein waves by Chang and Porkolab.[8] For plasma-heating purposes, of greater interest is the self-interaction of ion-Bernstein waves $\omega''(\mathbf{k}'')=-\omega(\mathbf{k})$, $\mathbf{k}''=-\mathbf{k}$, and $2\omega\simeq m\omega_{ci}$. For Bernstein waves for $m=3$, $k_\perp r_{ci}\simeq 1.5$–2.0 (the precise value depends on $k_\parallel$), $k_\perp'=2k_\perp\simeq(3\text{–}4)/r_{ci}$ and therefore $\omega'/k_\perp'\simeq(T_i/m_i)^{1/2}$ [here $r_{ci}=(T_i/m_i)^{1/2}\omega_{ci}$ is the ion gyroradius]. Therefore, we expect interaction and heating mainly with the bulk of the ion distribution.

The nonlinear equation governing ion-Bernstein wave propagation is[7]

$$\frac{\partial E_k}{\partial x}+\alpha_k E_k=\frac{L_{k,k''}|E_{k''}|^2E_k}{\partial\epsilon_R(\omega,k)/\partial k_\perp}, \tag{2}$$

where $\alpha_k=\epsilon_{\mathrm{Im}}/(\partial\epsilon_R/\partial k_\perp)$ is the linear spatial damping rate of the wave $\omega(\mathbf{k})$ in the x direction, and where the nonlinear matrix element is given by[6,7]

$$L_{k,k''}=\left(\frac{m\omega_{ci}}{k_{\parallel}'v_{ti}}\right)\left(\frac{\omega_{pi}^4}{\omega_{ci}^4}\right)\frac{V}{4\pi^{1/2}}\frac{\exp[-(\omega'-m\omega_{ci})^2/(k_{\parallel}'v_{ti})^2]}{n_0\kappa T_i}, \tag{3}$$

$$V=\sum_{s,p}\frac{A-BC/D}{[(\omega/\omega_{ci}-s)^2-1][(\omega/\omega_{ci}-p)^2-1]}.$$

Here

$$A=\int_0^\infty d\xi(\partial g/\partial\xi)J_s(z)J_p(z)J_{p-m}(z'')J_{s-m}(z''),\quad B=\int_0^\infty d\xi(\partial g/\partial\xi)J_p(z)J_m(z')J_{p-m}(z''),\quad C=B(p\to s),$$

$$D=\int_0^\infty d\xi(\partial g/\partial\xi)J_m^2(z')=-2I_m(b')e^{-b'},\quad g(\xi)=\exp(-\xi/2),\quad \xi=2v_\perp^2/v_{ti}^2,\quad v_{ti}^2=2T_i/m_i,$$

$$b_i'=k_\perp'r_{ci}^2,\quad z=k_\perp v_\perp/\omega_{ci},\quad z'=k_\perp'v_\perp/\omega_{ci},\quad z''=k_\perp''v_\perp/\omega_{ci}.$$

To analyze the self-interaction of the wave $\omega(k)$, we set $E_{k''} = -E_k$, and Eq. (2) may be rewritten as follows:

$$\partial E_k/\partial x + \alpha_k E_k + Q(x)|E_k|^2 E_k = 0. \tag{4}$$

In a uniform plasma $Q_0 = -L_{k,k''}/(\partial\epsilon_R/\partial k_\perp)$, and in a radially varying magnetic field

$$Q(x) = \frac{m\omega_{ci}}{2k_\parallel v_{ti}} \frac{(-V)}{4\pi^{1/2}} \frac{\omega_{pi}^4}{\omega_{ci}^4} \frac{\exp(-x^2/\zeta^2)}{(\partial\epsilon_R/\partial k_\perp) n_0 \kappa T_i} \equiv Q_0 \exp\left[\frac{-x^2}{\zeta^2}\right], \tag{5}$$

where

$$\zeta^{-1} = m\omega_{ci}(0)/(2k_\parallel v_{ti} L_B) = \omega/(k_\parallel v_{ti} L_B).$$

Here we expanded the magnetic field as $\omega_{ci}(x) \simeq \omega_{ci}(0)(1 - x/L_B)$; $x = 0$ is the resonant layer, L_B is the magnetic field gradient scale length in the x (i.e., radial) direction, and we kept the x dependence only in the exponential term. In a tokamak $L_B \simeq R$, the major radius, and Eq. (5) is valid for $x \leq R$. For ion-Bernstein waves $v_{te} << \omega/k_\parallel$, $k_\parallel^2 << k_\perp^2$, $v_{ti} << |\omega - m\omega_{ci}|/k_\parallel$, so that electron Landau damping and ion-cyclotron damping are weak. In a plasma with a Maxwellian distribution of electrons and ions the mode $(\omega, \mathbf{k})$ is determined by the dispersion relationship $\epsilon_R(\omega,k) = 0$, where[1,2]

$$\epsilon_R(\omega,k) = 1 + \frac{\omega_{pe}^2}{\omega_{ce}^2} - \frac{k_\parallel^2}{k^2}\frac{\omega_{pe}^2}{\omega^2} + \frac{1}{k^2\lambda_{Di}^2}\sum_j \theta_j \left[1 - \Gamma_0(b_j) - \sum_{l=1}^{\infty} \frac{2\omega^2\Gamma_l(b_j)}{\omega^2 - l^2\omega_{cj}^2}\right], \tag{6}$$

where the summation over j includes all ion species, $j = i$ designates the majority ion species, $j \neq i$ is the minority species, $\theta_j = n_j Z_j^2/n_i$, Z_j is the ionic charge state of ion species j, and typically $Z_i = 1$ so that $\theta_i = 1$. Further, $\Gamma_l = I_l(b_i)\exp(-b_i)$, $I_l(b_i)$ is the modified Bessel function of order l and argument $b_i = k_\perp^2 r_{ci}^2$, and the remaining notation is standard. One can show that $\partial\epsilon_R/\partial k_\perp$ in Eqs. (2) and (5) is given by

$$\partial\epsilon_R/\partial k_\perp = (2/k_\perp)(\omega_{pi}^2/\omega_{ci}^2) Y,$$

where

$$Y = \sum_j \theta_j \left(\frac{b_j}{b_i}\right)^{1/2} \left[\Gamma_0(b_j) - \Gamma_1(b_j) - \sum_{l=1}^{\infty} \frac{\omega^2[\Gamma_{l-1}(b_j) + \Gamma_{l+1}(b_j) - 2\Gamma_l(b_j)]}{\omega^2 - l^2\omega_{cj}^2} + \delta\right],$$

and where for $j = i$, $\theta_j(b_j/b_i)^{1/2} \equiv i$. Here

$$\delta = (1 + \omega_{pe}^2/\omega_{ce}^2)(1 + b_i)(\omega_{ci}^2/\omega_{pi}^2) \simeq 10^{-3}$$

is small and may be ignored.

The solution of the nonlinear Eq. (4) can be obtained by the pseudopotential technique outlined by Porkolab and Goldman in connection with the upper-hybrid soliton problem.[9] First consider the case of a uniform magnetic field. Ignoring temperature gradients, and assuming $E_k = \rho(x)^{1/2}\exp(i\sigma)$, one can integrate Eq. (4) and obtain

$$|E(x)|^2 = \frac{|E_a|^2 \exp(-2|\alpha_k|\Delta x)}{1 + (Q_0|E_a|^2/\alpha_k)[1 - \exp(-2|\alpha_k|\Delta x)]}, \tag{7}$$

where $\Delta x = (a - x) > 0$ is the total distance of propagation in the radial direction, and $E_a = E(a)$ is the amplitude of the launched wave at the plasma edge. For a linearly weakly damped pump wave $2\Delta x|\alpha_k| << 1$ (i.e., weak electron Landau damping is assured by taking $3 < \omega/k_\parallel v_{te}$), and one may expand the exponential factor:

$$|E(x)|^2/|E_a|^2 = [1 + 2\Delta x(|\alpha_k| + |Q_0||E_a|^2)]^{-1}.$$

Thus, the pump-wave power decreases to $1/e$ of its initial value when

$$2\Delta x|Q_0||E_a|^2 \gtrsim 1.72. \tag{8}$$

The pump-wave electric field $|E_a|$ may be estimated by considering electrostatic wave packets launched by an antenna of surface area A perpendicular to the x direction. The power carried by the wave packet is given by

$$P = \frac{|E_k|^2}{8\pi}\omega\frac{\partial\epsilon_R}{\partial\omega} v_{gx} S = \frac{|E_k|^2}{8\pi}\omega\frac{\partial\epsilon_R}{\partial k_\perp} S, \tag{9}$$

where S is the cross-sectional area of the wave packet normal to the x direction. When we combine Eqs. (5),(8), and (9) the condition for pump depletion is given by

$$\frac{\pi^{1/2}10^{19} b_i |V|(P/S)\Delta x}{2.75 r_{ci}^2 k_\parallel v_{ti} n_0 T_i Y^2} > 1, \tag{10}$$

where P/S is given in watts per square centimeter, T_i is in electronvolts, and all other quantities are in cgs units. Consider the center cell of a present-day tandem-mirror device such as the Livermore Tandem Mirror Experiment–Upgrade (TMX-U), and take $k_\parallel = (\pi/150) \simeq 0.02\ \text{cm}^{-1}$, $f = 4.5$ MHz, $f/f_{ci} = 1.5$, $T_i = 225$ eV, and $T_e = 50$ eV, and consider solely D^+ ions. The solution to Eq. (6) is $b_i \simeq 4.0$. A numerical integration of Eq. (3) with $m = 3$ and $k_\perp r_{ci} \simeq 2$ gives[10] $V \simeq 0.08$. For a uniform central plasma region of $2\Delta x = 15$ cm, condition (10) for pump depletion is

$$\frac{P}{S} \gtrsim \left(2.7\,\frac{\text{W}}{\text{cm}^2}\right) Y^2 \left(\frac{T_i}{225\ \text{eV}}\right)^{5/2} \left(\frac{n_0}{2\times 10^{12}\ \text{cm}^{-3}}\right),$$

where the ion temperature is normalized to $T_i = 225$ eV, and the density is normalized to $n_i = 2\times 10^{12}$ cm^{-3}. The evaluation of Y is dependent on the value of $k_\perp r_{ci}$ chosen (i.e., to $k_\parallel$). For the present conditions one obtains $Y \simeq 0.13$, and hence one finds $P/S \simeq 0.05\ \text{W/cm}^2$. If we consider an antenna of length 150 cm and height 10 cm, we have $A = 1.5\times 10^3$ cm^2, and for an effective area of $S \simeq 2A$ (to allow for wave spreading parallel to the magnetic field) the critical power is ~ 0.15 kW. Thus, for a typical injected rf power of 100 kW or more, this threshold is always exceeded. At a density of $n_0 \simeq 1\times 10^{13}\ \text{cm}^{-3}$ and $T_i \simeq 1$ keV, the critical power density would remain near 8 W/cm^2, or 24 kW total power for an effective cross-sectional area $S \simeq 2A = 3\times 10^3\ \text{cm}^2$ (spreading by more than a factor of 2 in TMX-U is not expected since the axial magnetic field gradients beyond the antenna reflect the wave packets).

Let us now consider a form of $Q(x)$ given by Eq. (5), which may be relevant to a tokamak or a bumpy-torus geometry. Taking the trial solution

$$E_k = \rho(x)^{1/2} \exp[i\sigma + x^2/2\zeta^2],$$

one finds that $\sigma = \text{const}$ and

$$d\rho/dx + 2\rho(\alpha_k + \rho Q_0 + x/\zeta^2) = 0.$$

This equation can be integrated by first changing variables to $Z = 1/\rho$. Then the integral can be performed, and after inversion of variables the result is

$$|E(x)|^2/|E_a|^2 = \{1 + \pi^{1/2}\zeta_i |Q_0||E_a|^2[1 - \text{erf}(x/\zeta_i)]\}^{-1}, \quad (11)$$

where $\text{erf}(x/\zeta_i)$ is the error function, and where the linear damping (α_k) has been neglected. Note that significant nonlinear interaction occurs only where $x^2 \lesssim \zeta^2 << a$, and therefore we replaced the lower limit of integration $x = a$ with $x = \infty$.

The total absorption across the plasma cross section ($\Delta x = 2a$) is

$$|E(-a)|^2/|E_a|^2 = [1 + 2\pi^{1/2}\zeta_i |Q_0||E_a|^2]^{-1}.$$

Thus, the critical power for pump depletion is $2\pi^{1/2}\zeta_i|Q_0||E_a|^2 \gtrsim 1.72$. With substitution for ζ_i, Q_0, and $|E_a|^2$ in terms of the incident rf power density, the threshold for pump depletion is

$$\frac{\pi 10^{19}}{2.75}\,\frac{b_i}{r_{ci}^2}\,\frac{L_B|V|(P/S)}{\omega n_c T_i Y^2} \geqslant 1, \quad (12)$$

where P/S is again measured in watts per square centimeter, T_i is measured in electronvolts, and other quantities are in cgs units. As expected, the absorption increases with $L_B \simeq R$, the major radius of the tokamak. For the Nagoya JIPP-II-U parameters,[4] namely $B = 1.8$ T, $n_0 \simeq 2\times 10^{13}\ \text{cm}^{-3}$, $R = 91$ cm, $T_i \simeq 300$ eV, $T_e = 700$ eV, $f = 40$ MHz, $k_\parallel \lesssim 0.05$, and $\omega/\omega_{ci} = 1.50$, one finds $b_i \simeq 1.50$, $Y \simeq 0.27$. Numerical integration[10] of Eq. (3) gives $V \simeq 0.02$ for $m = 3$, and Eq. (12) yields $P/S \gtrsim 35\ \text{W/cm}^2$. Taking an antenna area of $A \simeq 300\ \text{cm}^2$, for $S \simeq 2A$ one obtains a critical pump power of $P \simeq 20$ kW, which is within the experimental value of $P \lesssim 100$ kW, and is in reasonable agreement with the apparent threshold power observable in Ref. 4, Fig. 4.

Wave absorption in the case of $f/f_{ci}(0) = 2.5$, $m = 5$, was also examined. Taking $f = 80$ MHz for the Nagoya JIPP II-U parameters one would obtain $k_\perp r_{ci} \simeq 2.0$, $Y \simeq 0.3$, $V \simeq 0.02$, and the threshold power would remain nearly the same as for the case of $m = 3$.

In the presence of impurities (minority species) with $Z_j/m_j = Z_i/2m_i$, $k_\perp r_{ci}$ would increase significantly and $\partial\epsilon_R/\partial k_\perp$ (i.e., Y) would greatly be reduced. These effects would result in significant reduction of the critical power for nonlinear pump-wave depletion. Furthermore, because of their heavier ion masses $\zeta_j = k_\parallel v_{tj} L_B/\omega$, the width of the impurity-ion cyclotron resonance is considerably smaller than ζ_i, that of the majority (hydrogen) ions. Therefore, for sufficiently strong electric fields complete nonlinear pump depletion may occur before the wave arrives at the minority cyclotron absorption layer. To quantify these concepts, the nonlinear electric field, Eq. (11), should be compared with that attenuated by linear ion cyclotron damping by impurities, namely

$$E^2(x)/E^2(\infty) = \exp\left(-2\int_\infty^x k_{IM}\,dx\right),$$

where $k_{IM} = \epsilon_{IM}(\text{impurity})(\partial\epsilon_R/\partial k_\perp)^{-1}$. After integration one obtains

$$E(x)^2/E(a)^2 = \exp\{-\tfrac{1}{2}\pi[\Gamma_m(b_j)\theta L_B/r_{ci} b_i^{1/2} Y][1 - \text{erf}((x/\zeta_i)(m_j/m_i)^{1/2})]\}, \quad (13)$$

where $\theta_j = n_j Z_j^2/n_i$, or in terms of Z_{eff}, the effective ion charge, $\theta_j = Z_j(Z_{\text{eff}} - 1)/(Z_j - Z_{\text{eff}})$. The predictions of Eqs. (11) and (13) were compared for several cases of interest for the JIPP II-U parameters. For example, assuming $n_{He}/n_e = 0.014$, one finds $b_i \simeq 2.0$, $Y = 0.02$, $V \simeq 0.06$, and at $x/\zeta_i = 1.0$ nonlinear pump depletion [Eq. (11)] requires $P/S \gtrsim 0.06$ W/cm^2, a very low value (i.e., for $A \simeq 2S \sim 600$ cm^2, $P \simeq 36$ W). Therefore, in the experiments where $P \gtrsim 10$ kW, nonlinear absorption will occur at $x/\zeta_i > 1$. On the contrary, at $x/\zeta_i = 1.0$ linear damping [Eq. (13)], would produce $E^2(x)/E^2(a) \simeq 0.98$, or only minor wave-energy loss (even though at $x \rightarrow 0$, complete linear pump-wave absorption would result). As a second example, we examined the case of $n_{C^{6+}}/n_e \simeq 0.014$ which would yield $Z_{eff} = 1.42$. The result was even more dramatic than for $^4He^{2+}$. In this case again complete nonlinear pump depletion resulted at $x/\zeta_i > 1.0$ at power levels $P/S \lesssim 1$ W/cm^2, whereas linear absorption at this position was completely negligible since at $x/\zeta_i = 1.0$, $(x/\zeta_i)(m_j/m_i)^{1/2} = 3.46$. Similar results follow at either smaller (or larger) concentrations of C^{6+}, O^{8+}, or other similar minority ions.

In summary, it has been shown that nonlinear ion Landau (cyclotron) damping efficiently absorbs the pump-wave power during ion-Bernstein–wave heating experiments in present-day tokamaks and tandem mirrors. In the presence of impurity (minority) ions this nonlinear majority-ion absorption is even stronger and completely dominates linear cyclotron damping by impurities because of the latter's narrower width of resonance.

After this manuscript was submitted, the paper by Abe *et al.*[11] predicting the $\omega/\Omega_H = \frac{3}{2}$ nonlinear resonance absorption was published. This paper shows partial nonlinear absorption of the wave based on limited particle-simulation code results. Further, the simulation results are interpreted in terms of a single-particle, single-wave force equation which does not correctly include collective (plasma shielding) effects.

Further, the paper by Toi *et al.*[12] presented new experimental evidence for the importance of nonlinear absorption in the JIPP T-IIU experiments.

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