Analytical Model for Large-Scale Turbulence

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We present an analytical model for large-scale turbulence, based on a new closure which depends on the growth rate of the instability-generating turbulence. For convection in stars we recover the results of the mixing-length theory and for laboratory convection we recover the $N \sim R^{1/3}$ law. The present model can readily be extended to include magnetic fields and rotation.

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Despite the fact that large-scale turbulence (LST) is responsible for bulk properties of direct astrophysical and geophysical interest like energy fluxes and temperature and velocity fluctuations, no analytical model for LST has yet been proposed which can claim a success comparable to that of the Heisenberg-Kolmogoroff (HK) model[1] for small-scale turbulence (SST). The proof[2] that extending the HK model to large-scale turbulence leads to unphysical results, as well as the LST dependence on the specific nature of the stirring mechanism, have led some[3] to conclude that the analytical modeling of LST is "formidable in the extreme." As a consequence, there has been an increased effort in numerical simulations[4] of LST, an approach which, however, cannot be expected to cover the parameter space of interest in astrophysics and geophysics. In this paper we propose an analytical model for LST, compare its implications with astrophysical and laboratory turbulent convection, and discuss future generalizations.

Following Ledoux, Schwarzschild, and Spiegel[5] and Yamaguchi,[5] we shall write the energy balance equation as $E_{\text{gain}} = E_{\text{loss}}$, i.e.,

$$2\int_{k_0}^{k} n_s(k)F(k)dk = 2\nu_t(k)\int_{k_0}^{k} k^2F(k)dk. \tag{1}$$

Here, $F(k)$ is the energy spectral function [$F(k)dk$ represents the turbulent energy in the wave-number interval between k and $k+dk$], $n_s(k)$ is the growth rate of the instability that generates turbulence, $\nu_t(k)$ is an eddy viscosity, and k_0 is the smallest wave number compatible with the size of the system.

Equation (1) comprises the following processes: energy gain from buoyancy forces, energy loss by kinematic viscosity ν and heat conduction χ, energy gains or losses due to external fields, and finally, energy losses or gains due to dynamical nonlinear interactions. The first set of processes can be accounted for by the linear theory; their net effect is represented by the growth rate $n_s(k)$ and the net energy gain E_{gain} is given by the left-hand side of Eq. (1), where the factor of 2 arises because the energy is a quadratic function of the amplitude.

The last process, represented by E_{loss}, cannot be accounted for by the linear theory and is written as a two-step process: first, the removal of energy from the interval k_0–k (written in analogy to molecular viscosity), followed by the redeposition of the same energy into the remaining interval from k to ∞. The latter is represented by an eddy viscosity $\nu_t(k)$ exerted upon modes of wave number k by modes with higher k's, i.e.,

$$\nu_t(k) = \int_{k}^{\infty} \nu_t^{(k)} dk/k, \tag{2}$$

where $\nu_t^{(k)}$ is the eddy viscosity exerted by turbulence on a band of wave numbers centered around k. In order to know $F(k)$ in the interval k_0–k, one must therefore know the eddy viscosity exerted by eddies in the interval k–∞. The function $\nu_t^{(k)}$ is, however, not known (the closure problem). By selecting a band of wave numbers with $k \gg k_0$, i.e., small eddies sufficiently removed from k_0 as to behave independently of the nature of the energy source, Heisenberg and Kolmogoroff proposed an expression for $\nu_t^{(k)}$ that yields a spectral function $F_{\text{HK}} \sim k^{-5/3}$ in agreement with data on small-scale turbulence. However, the range of validity of the HK closure makes it unsuitable for the estimate of bulk turbulent properties which are contributed mostly by large-scale eddies. The convective fluxes computed in Refs. 5 using HK closure do not agree well with the results of the mixing-length model, Eq. (11).

In this paper, we are interested in deriving the spectral function $F(k)$ appropriate for large-scale turbulence, i.e., for wave numbers close to k_0. To accomplish that, we must first know the form of $\nu_t^{(k)}$ in almost the entire wave-number interval, $k_0 \lesssim k \lesssim \infty$, a requirement clearly more demanding than in the HK case. The disadvantageous situation can be turned around by noting that in spite of not knowing the full form of $\nu_t^{(k)}$, we do know the integral of it over the entire k range (k_0–∞). In fact, for any $\nu_t(k)$, Eq. (1) implies that

$$\nu_t(k_0) = n_s(k_0)/k_0^2, \tag{3}$$

which allows us to rewrite (2) as

$$\nu_t(k) = \nu_t(k_0) - \int_{k_0}^{k} \nu_t^{(k)} dk/k. \tag{4}$$

The problem is then reduced to that of finding $\nu_t^{(k)}$ in the interval k_0–k. Equation (4) has the further advantage that it allows us to test the validity of the expression for $\nu_t^{(k)}$ that we want to propose. *If* this suggested $\nu_t^{(k)}$ were correct for the entire k_0–∞ interval, it would satisfy the integral property (3). [See the comments after Eq. (8).]

To construct $\nu_t^{(k)}$, consider that in general[6] $\nu_t^{(k)} = \lambda_k u_k$, $\lambda_k = \tau_k u_k$, $u_k = (kF)^{1/2}$, where λ_k is an effective mean free path, u_k a typical turbulent velocity, and τ_k a correlation time. Calling $\tau_k \equiv n_c^{-1}$, we have

$$\nu_t^{(k)} = kF(k)/n_c(k). \tag{5}$$

Equation (5) is of general validity. In the HK model, where the eddies are freely evolving, $\lambda_k = l_k \sim k^{-1}$ and so $n_c^{\mathrm{HK}} \sim k^{3/2}F^{1/2}$. However, since the large eddies are strongly forced upon by the stirring mechanism, their correlation time τ_k can no longer be simply identified with the turnover time.

Substituting (5) and (4) into (1), one obtains a nonlinear integral equation for $F(k)$ which can be solved. The result is (the prime denotes d/dk)

$$-2F(k) = k^{-2}\{kn_c^{1/2}\int_{k_0}^{k} kn_c^{1/2}(n_s k^{-2})' dk\}'. \tag{6}$$

In order to specify n_c, we shall make use of two independent arguments. First, the creation of a generic eddy is contributed both by the breakup of larger eddies as well as the growth of the instability-generating turbulence. For the largest eddies, $k \approx k_0$, only the latter process is operative, thus suggesting $n_c(k_0) \propto n_s(k_0)$. We propose to extend this result to the entire group of LST eddies, and take

$$n_c(k) = \frac{n_s(k)}{n_s(k_0)} n_c(k_0) \equiv \gamma^{-1} n_s(k). \tag{6a}$$

The second argument is based on recent work on differentially rotating disks.[7] It can be shown that the component of the Reynolds stress tensor τ_{ij} acting on the mean azimuthal rotation can be expressed as $\tau_{r\phi} = \nu_t^* r\Omega'$, where Ω is the angular velocity. The turbulent viscosity ν_t^* can be shown[7] to be given by $\sim \int F n_s^{-1} dk$. Since it is physically reasonable to expect that ν_t^* is roughly equal to the turbulent viscosity exerted on the largest eddies, we are led to the choice (6a) for $n_c(k)$. It is, however, important to stress that (6a) applies only to the LST region, where $n_s(k) > 0$, and not to the region in k space where $n_s(k)$ may become negative.

Before we apply the above results, we need γ, i.e., $n_c(k_0)$. We do so by requiring $\lambda(k_0) = L_p$, where L_p is the longitudinal integral scale defined in Eq. (2.51) of Orszag.[1] Since $\lambda(k_0)$ depends on $F(k_0)$ which in turn depends on $n_c(k_0)$, we obtain

$$-2n_c(k_0) = k_0 L_p^{-2}(n_s k^{-2})'_0. \tag{6b}$$

Convection.—In the case of thermally driven convection,[8] the linear growth rate $n_s(k)$ is given by ($k = k_0 q$)

$$n_s(k)/n_0 = [1 + (1-\mu)\lambda^2 q^4]^{1/2} - \lambda q^2, \tag{7}$$

where $2n_0 = \chi(1+\sigma)k_0^2\lambda^{-1}$ and $\lambda^2 = (1+x)^3\pi^4 \times (x\mu R)^{-1}$. Here, χ is the thermometric conductivity, $\sigma = \nu/\chi$ the Prandtl number, and R the Rayleigh number $= g\alpha\beta d^4/\nu\chi$; d is the depth of the convective layer, α is the thermal expansion coefficient, g is the local gravity, β is the temperature gradient excess over the adiabatic gradient, and $\mu = 4\sigma(1+\sigma)^{-2}$. The parameter $x = (k_x^2 + k_y^2)/k_z^2$ represents the degree of anisotropy in the eddies sizes. As in previous work,[5] $k_0 d = \pi(1+x)^{1/2}$.

Once (7) is substituted in (6), the integration can be performed analytically for $\mu << 1$, in which case $n(k) > 0$ for all k's. The resulting $F(k)$ is

$$F = F_0 q^{-2} n^{3/2}(1+\lambda^2 q^4)^{-1/2}, \tag{8}$$

where $F_0 = C_\lambda n_0^{1/2} k_0^{-3}\gamma^{-1}$ and $C_\lambda^2 = (1+\lambda^2)^{1/2} + \lambda$. For $\lambda q^2 << 1$, $F \sim k^{-2}$, while for $\lambda q^2 >> 1$, $F \sim k^{-7}$, respectively. Using (7) we derive $\nu_t(k_0) = \nu(1+\sigma^{-1}) \times (2\lambda C_\lambda^2)^{-1}$. On the other hand, if we use (5), (7), and (8) in (2) and take the limit $k \to k_0$, i.e., if we assume that our closure is valid for the entire k interval, we obtain $\nu(1+\sigma^{-1})(C_\lambda/2\lambda)[C_\lambda - (2\lambda)^{1/2}]$ which as expected does not coincide with the exact expression for $\nu_t(k_0)$ just derived. It is, however, interesting that the two results do coincide in the case of $\lambda << 1$, which obtains in most astrophysical cases. Using $n_s(k)$ and $F(k)$, we have evaluated convective fluxes, F_c, and temperature and velocity fluctuations which can be given in analytical form. However, since the most reliable data are for convective fluxes, we shall present the results only for $F_c = c_p\rho\beta\chi\Phi$, where it can be shown that

$$\Phi = \frac{1}{g\alpha\beta\chi}\int_{k_0}^{\infty}[n_s(k) + \nu k^2]F(k)dk. \tag{9}$$

Performing the integration, we derive

$$\Phi = \frac{1}{\gamma}\frac{x}{1+x}(2\lambda)^{-1}\{[C_\lambda - (2\lambda)^{1/2}]^2 + \sigma C_\lambda[C_\lambda - (2\lambda)^{1/2}]\}. \tag{10}$$

Using (7) and (8) in (6b), we obtain for γ, with $\omega \equiv (2\lambda)^{1/2}C_\lambda$,

$$\gamma = (3\pi l/4)^2(1-\lambda C_\lambda^{-2}), \quad 2(1-\omega\tan^{-1}\omega^{-1})l = 1 - 3\lambda C_\lambda(\tanh^{-1}C_\lambda^{-1} - \tan^{-1}C_\lambda^{-1}). \tag{10a}$$

Astrophysics.—In most astrophysical settings (convective layers of stars, accretion disks, etc.), σ is much smaller than unity. Equation (10) can then be rewritten as

$$\Phi = aS^{-1}[(1+bS)^{1/2}-1]^3, \tag{11}$$

where $S=\sigma R$, $8a\gamma=(1+x)^2\pi^4 A^2(\lambda)$, and $b=4x\times(1+x)^{-3}\pi^{-4}$. The function $A^{-1}(\lambda)=1+(2\lambda)^{1/2}\times C_\lambda^{-1}$ is almost independent of λ: For $\lambda \ll 1$, $A=1$, while for $\lambda \gg 1$, $A=\frac{1}{2}$. The structure of Eq. (11) is identical to that proposed by Bohm-Vitense[9] using the mixing-length theory. Furthermore, the dependence of (11) on the anisotropy factor x is identical to the one proposed on the basis of a phenomenological model for the eddies annihilation probability.[10] Furthermore, we have verified that the expression for the turbulent velocity is identical to the one derived from the mixing-length model.[9]

Laboratory turbulent convection.— A major goal of experimental and theoretical research on turbulent convection[8, 11–14] has been the search for the relation between the Nusselt number N and the Rayleigh number $R_{\exp}=g\alpha\Delta T D^3/\nu\chi$, where D is the distance between the two plates and ΔT the temperature drop. Experiments performed with different substances were found to exhibit a dependence of the type $N\sim R_{\exp}^b$ with $0.257\leqslant b\leqslant 0.333$. Recently, Goldstein and Tokuda,[12] performing experiments with $R_{\exp}\leqslant 2\times 10^{11}$ and $5.71\leqslant\sigma\leqslant 6.76$, established the relation

$$N=0.0556R_{\exp}^{1/3}. \tag{12}$$

The confirmation of the $\frac{1}{3}$ power is of major importance since this value has long been suggested to be the consequence of a strongly distorted temperature profile, whereby $\beta(z)=dT/dz$ is nonzero *only* in two thin boundary layers (zones I and III) near the two plates, each of thickness L, while in the largest part (zone II) the fluid is nearly isothermal, $\beta(z)\simeq 0$. The total flux $H=\chi\Delta T D^{-1}N$, being controlled by the two boundary layers, is then independent of D. This implies $N\sim R_{\exp}^{1/3}$.

Because of the general nature of this argument, the N vs R relation has been studied with use of different models.[12, 13] However, we are not aware of a derivation of (12) from a model of turbulence. Given the division into three regions I, II, and III, we have solved the linear equations[8] for the temperature θ and the z component of the velocity ω separately in the three regions, assuming solutions of the form $\sim\sin[(z,D-z)k]$ in I and III where $\beta=$ const and corresponding hyperbolic functions in region II where $\beta=0$. Matching ω and ω' at $z=L$ and $z=D-L$, and using $2L\beta=D\beta_0$, where β_0 stands for $\Delta T/D$, we obtain ($\delta=k_zL$)

$$-\cot\delta=x^{1/2}\tanh[\delta x^{1/2}(\beta/\beta_0-1)]\simeq x^{1/2}, \tag{13}$$

where the second equality is because only large temperature distortions $\beta/\beta_0\gg 1$ are of interest. Furthermore, in regions I and III, $n_s(k)$ has the form (7) with λ^2 given in terms of $x, \beta, \ldots$ by

$$\lambda^2=16(1+x)^3(\beta/\beta_0)^3(x\mu R_{\exp})^{-1}\delta^4. \tag{14}$$

Finally, averaging the total flux over region I (or III), we obtain for the Nusselt number

$$N=(\beta/\beta_0)(1+\Phi), \tag{15}$$

where Φ is provided by (10). Since in laboratory convection (unlike in stars) D and ΔT are set experimentally, the energetically most favorable state is the one that requires the minimum energy input from the source to sustain a given ΔT. This is obtained by requiring that N be minimal with respect to β, i.e., we demand that $\partial N/\partial\beta=0$. Furthermore, we propose to fix x by maximizing N with respect to x. Using (14), (15), and (10), we obtain (here, prime denotes $d/d\lambda$)

$$1+\Phi+\tfrac{3}{2}\lambda\Phi'=0, \quad \Phi+\tfrac{1}{2}\lambda(2x-1+2x^{1/2}\delta^{-1})\Phi'=0, \tag{16}$$

which determine the values of x and λ. Knowing λ and x, we can use (14) to express β in terms of $R_{\exp}$. Substitution in (15) yields [with use of (16) to eliminate Φ]

$$N=A_\sigma R_{\exp}^{1/3}, \tag{17}$$

where

$$A_\sigma=\tfrac{3}{4}(1+x)^{-1}(x/2)^{1/3}\delta^{-4/3}(2-x-x^{1/2}\delta^{-1})^{-1}(\lambda^2\mu)^{1/3}. \tag{18}$$

Before we solve (16) for a given σ, some general results can be derived. (A) Using (16) and (10), one can prove that $0.44\leqslant x\leqslant 1.07$ and $\frac{1}{5}\leqslant\Phi\leqslant 2$. (B) In the limits $\sigma\to 0$ and $\sigma\to\infty$, we obtain

$$A_\sigma\sim\sigma^{1/3} \quad (\sigma\to 0),$$
$$A_\sigma=0.078\gamma^{-1/3} \quad (\sigma\to\infty). \tag{19}$$

Both results are in agreement with observations. In particular, the latter result is in agreement with the well-known[14] upper limit for N.

Solving Eqs. (16) with (10) and (10a) for $\sigma=6.8$ (water), we obtain

$$x=0.78, \quad \lambda=0.32, \quad \gamma=2.6, \quad A_\sigma=0.044, \tag{20}$$

to be compared with the measured value of $A_\sigma=0.0556$. Our model is therefore capable of reproducing 80% of the observed flux, a very encouraging result considering the simplicity of the model (part of the discrepancy may be due to the approximation $\mu \ll 1$, while for water $\mu = 0.45$). We have also studied the case $\sigma = 1000$ for which $\mu \ll 1$ is well satisfied. The results are

$$x = 0.65, \quad \lambda = 6.9, \quad \gamma = 2.05, \quad A_\sigma = 0.061. \quad (21)$$

In summary, we have presented a model for large-scale turbulence which (1) uses as the only ingredient the growth rate $n_s(k)$ of the instability, (2) is based on a new closure that depends on $n_s(k)$ itself, (3) is analytical, (4) in the case of astrophysics, reproduces the results of the mixing-length theory, i.e., the formula of Bohm-Vitense, and (5) in the case of laboratory turbulent convection, reproduces the recently established $N = AR^{1/3}$ relation and (6) predicts an N vs R relation in agreement with the general result derived by Howard.[14]

The present model can be easily extended to include the effects of rotation and/or magnetic fields.

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