Physical *CP* Phase and Maximal *CP* Nonconservation

M. Gronau[(a)]
Group T, Los Alamos National Laboratory, Los Alamos, New Mexico 87545

and

J. Schechter
Physics Department, Syracuse University, Syracuse, New York 13210
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We give some applications for a *CP*-nonconserving quantity which is invariant under nontrivial rephasings of the Kobayashi-Maskawa mixing matrix. It is shown to resolve one of the two ambiguities involved in defining the concept of maximal *CP* nonconservation. The second, more intrinsic, ambiguity is partially resolved in the framework of an especially symmetrical class of parametrizations.

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It would clearly be rather beautiful if nature chooses nonconservation of *CP* in a "maximal" way as she does for *P* and *C* separately. Although the characteristic ratio of *CP*-nonconserving to *CP*-conserving weak-decay amplitudes is around 10^{-3}, the *CP* nonconservation might be considered maximal if it were found to be associated with a certain parameter in the Lagrangian becoming as large as it could possibly be. Indeed, in the "standard" model of elementary particles all the Lagrangian-level *CP* nonconservation occurs in a single phase of the Kobayashi-Maskawa[1] (KM) mixing matrix. A great deal of experimental and theoretical study indicates[2] that the only way in which the standard model is likely to be consistent with experiment is for the sine of this phase to be large. Hence it seems there is some possibility that maximal *CP* nonconservation might hold as has been recently emphasized by several authors.[3,4]

This brings to the forefront a conceptual point. Since the KM matrix may be parametrized with possible *CP*-nonconserving phases in many different ways, how can we be sure that what we call maximal *CP* nonconservation is not just an artifact of a particular convention. In fact it has been questioned[5] whether there exists any reasonable way at all to state that this matrix leads to maximal *CP* nonconservation. There are actually two types of ambiguity. First, for a given presentation of the mixing matrix one can obtain new parametrizations by rephasing transformations. We will show that this ambiguity can be completely overcome by introducing a certain *invariant* phase. This concept will also be shown to be useful when considering the extension of the mixing matrix to four or more generations. The second type of ambiguity concerns the way in which the "angular" dependence of the mixing matrix is initially presented, e.g., how it is built up as a product of two-dimensional matrices. This ambiguity is in a sense intrinsic. For *each* presentation there is a clear definition of maximality, namely when the appropriate invariant phase is $\pi/2$. The real question is, of course, what underlying theory is appropriate to a given presentation. In other words, if the underlying theory is indeed maximally *CP* nonconserving which of the various presentations would demonstrate it with a phase of $\pi/2$. It at first might seem that nothing more could be said at the level of the mixing matrix. However, we shall see that if the (aesthetic) demand is made that the presentation not distinguish one generation from another, there are six natural parametrizations. These divide, for practical purposes, into just two equivalence classes. One of these classes furthermore can be ruled out experimentally as a candidate for a maximally *CP*-nonconserving theory.

The mixing matrix, U, in the standard model is a 3×3 unitary matrix which may be taken to be unimodular. It has a gaugelike freedom in that all physical predictions are unchanged when we multiply on the left-hand side by $\operatorname{diag}\{\exp[i(\alpha_1-\gamma_1)], \exp[i(\alpha_2-\gamma_2)], \exp[i(\alpha_3-\gamma_3)]\}$ and on the right-hand side by $\operatorname{diag}[(\exp-i\alpha_1), \exp(-i\alpha_2), \exp(-i\alpha_3)]$, where $\sum_i\alpha_i=\sum_i\gamma_i=0$. Originally,[1] a "gauge" was chosen by using this freedom to eliminate four phases from U (leaving four parameters). Here, however, we shall keep the two nontrivial extra phases.

For our purpose it is convenient[6,7] to build up U as a product of "complex rotations" connecting each possible pair of generations. The rotation ω_{12} between the first and second generations, for example, is given by

$$\omega_{12}=\begin{bmatrix} \cos\theta_{12} & e^{i\phi_{12}}\sin\theta_{12} & 0 \\ -e^{-i\phi_{12}}\sin\theta_{12} & \cos\theta_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \qquad (1)$$

θ_{12} expresses the amount of mixing between the first two generations while ϕ_{12} is a *CP*-nonconserving phase. The complex rotations ω_{ij} for general i and j are defined analogously. Then we choose as our standard form U,

$$\omega_{23}\omega_{12}\omega_{13}=\begin{bmatrix} c_{12}c_{13} & s_{12}e^{i\phi_{12}} & c_{12}s_{13}e^{i\phi_{13}} \\ -[c_{23}s_{12}c_{13}e^{-i\phi_{12}}-s_{13}s_{23}e^{i(\phi_{23}-\phi_{13})}] & c_{12}c_{23} & [-c_{23}s_{12}s_{13}e^{i(\phi_{13}-\phi_{12})}+s_{23}c_{13}e^{i\phi_{23}}] \\ [s_{23}s_{12}c_{13}e^{-i(\phi_{12}+\phi_{23})}-c_{23}s_{13}e^{-i\phi_{13}}] & -s_{23}c_{12}e^{-i\phi_{23}} & [c_{23}c_{13}+s_{23}s_{12}s_{13}e^{i(\phi_{13}-\phi_{12}-\phi_{23})}] \end{bmatrix}, \quad (2)$$

where $c_{ij}=\cos\theta_{ij}$ and $s_{ij}=\sin\theta_{ij}$. Equation (2) differs from the most general unitary 3×3 matrix by the trivial possibility of multiplication by a diagonal matrix of phases on the left. We have eliminated these trivial phases with the parameters γ_i. Note, by Eq. (2), that this is equivalent to taking the diagonal (11) and (22) elements of U to be real. We thus will always consider U to remain unimodular with two real diagonal elements. In section II of Ref. 7 it is shown that the reparametrization due to the nontrivial phases α_i may be simply described by the transformation

$$\phi_{ij} \to \phi'_{ij}=\phi_{ij}+\alpha_i-\alpha_j. \quad (3)$$

To find a CP-nonconserving quantity which is independent of the phase-related parametrizations of U we look for an object which is invariant under (3). This object is evidently (defining $\phi_{31}=-\phi_{13}$)

$$\Phi=I_{123}=\phi_{12}+\phi_{23}+\phi_{31}. \quad (4)$$

It then seems natural to choose as the condition for maximal CP nonconservation the following:

$$\Phi=\tfrac{1}{2}\pi\times(\text{odd integer}). \quad (5)$$

Notice that both of the explicit examples of U used in Ref. 5 to illustrate a possible ambiguity in the definition of maximal CP nonconservation are special cases[8] of Eq. (2). Thus if (5) is used as the condition for maximal CP nonconservation there is no ambiguity.

It is easy to generalize the invariant phase to any number of generations. For the case of four generations the following choice of U was recently noted[9] to be very convenient:

$$U=[\omega_{34}\omega_{24}\omega_{14})(\omega_{23}\omega_{12}\omega_{13}). \quad (6)$$

In addition to I_{123} the three combinations

$$I_{124}=\phi_{12}+\phi_{24}-\phi_{14},$$

$$I_{134}=\phi_{13}+\phi_{34}-\phi_{14},$$

$$I_{234}=\phi_{23}+\phi_{34}-\phi_{24}, \quad (7)$$

are invariant under rephasings of U. Since

$$I_{123}-I_{124}+I_{134}-I_{234}=0, \quad (8)$$

there are, as expected, only three independent CP-nonconserving invariants in the four-generation case. This approach makes manifest the fact that the three vanishing phases cannot be chosen to be ϕ_{12}, ϕ_{23}, and ϕ_{13} since that would force the *arbitrary* invariant I_{123} to be zero. Physically, this means that the three phases connecting the first three generations cannot be swept away, as one might initially expect on the basis of counting alone.

As another application of the invariant phase, notice that since in (2) each phase $\exp(\pm i\phi_{ij})$ is multiplied by s_{ij}, Φ must always appear in a combination $s_{12}s_{13}s_{23}e^{i\Phi}$. This has the consequence that *all CP-nonconserving physical quantities will be proportional to* $s_{12}s_{13}s_{23}\sin\Phi$. A similar result was obtained by explicit computation by Chau and Keung.[10] Note that for the unimodular matrix with two real diagonal elements in Eq. (2) this product is simply $-\operatorname{Im}\operatorname{Tr}U$. A convenient feature of the parametrization (2) is that each angle may be approximately identified (within several percent) with the magnitude of a physical quark transition amplitude, i.e., $s_{12}=|U_{us}|$, $s_{13}\simeq|U_{ub}|$, $s_{23}\simeq|U_{cb}|$. Thus within this approximation all CP-nonconserving quantities contain an overall factor $|U_{us}U_{ub}U_{cb}|\sin\Phi$.

The presentation (2) is symmetric in the sense that each generation is being treated on the same footing by factoring U into "complex rotations" in each of the three planes. However, there is an ambiguity as to which order of multiplication is to be used. There are five other symmetrical presentations using ω_{12}, ω_{23}, and ω_{13} in different orders. These may be related to (2) in the approximation[11] that $s_{12}=O(\epsilon)$, $s_{23}=O(\epsilon^2)$, and $s_{13}=O(\epsilon^3)$, with small ϵ. The presentations[12] $\omega_{23}\omega_{13}\omega_{12}$ and $\omega_{13}\omega_{23}\omega_{12}$ are then the same as (2) while the presentations $\omega'_{13}\omega'_{12}\omega'_{23}$, $\omega'_{12}\omega'_{13}\omega'_{23}$, and $\omega'_{12}\omega'_{23}\omega'_{13}$ are equivalent to each other but different from (2). The relation between the two classes is given by $s'_{12}=s_{12}$, $\phi'_{12}=\phi_{12}$, $s'_{23}=s_{23}$, $\phi'_{23}=\phi_{23}$, $\tan\Phi'=\sin\Phi[\cos\Phi-s_{12}s_{23}/s_{13}]^{-1}$, $s'_{13}=s_{13}\sin\Phi/\sin\Phi'$. If we consider $\Phi'=\phi'_{12}+\phi'_{23}-\phi'_{13}$ to be $\pm\pi/2$ we obtain $|\cos\Phi|\simeq|U_{us}U_{cb}/U_{ub}|$, which is greater than one experimentally. Thus this alternative class of presentations is ruled out as a possibility for defining maximal CP nonconservation.

We are specifying U by three angles and three phases in all the parametrizations above. It is interesting to notice that a unimodular U with two real diagonal elements may be specified (but possibly not neatly parametrized) directly by the magnitudes and phases of (U_{12},U_{13},U_{23}) or of (U_{21},U_{31},U_{32}). Writing $U_{ij}=|U_{ij}|\exp(\chi_{ij})$ we may easily see that

$\chi=\chi_{12}+\chi_{23}-\chi_{13}$ and $\chi'=\chi_{21}+\chi_{32}-\chi_{31}$ are invariant phases. The (U_{12},U_{13},U_{23}) specification is evidently equivalent to (2) while the (U_{21},U_{31},U_{32}) specification may be seen, from the remark in footnote 12, to be of the primed class above. The quark mixing matrix, in the presentations equivalent to (2) may be approximately written in terms of the experimental transition amplitudes:

$$U \simeq \begin{pmatrix} 1 & |U_{us}| & |U_{ub}|e^{i\Phi} \\ -|U_{us}| & 1 & |U_{cb}| \\ |U_{cb}U_{us}|-|U_{ub}|e^{-i\Phi} & -|U_{cb}| & 1 \end{pmatrix}, \tag{9}$$

where we have rotated away ϕ_{12} and ϕ_{23}.

It seems perhaps less likely that unsymmetrical presentations, which do not treat each generation on an equal footing, would naturally describe an elegant underlying theory. However, these may be discussed in a similar way to the symmetrical ones. Here we shall only consider the original KM choice.[1] This is essentially of the "Euler" type $\omega_{23}\omega_{12}\omega'_{23}$. (An invariant phase is $\phi_{23}-\phi'_{23}$.) The original KM matrix may be put in our standard form (2) by a trivial phase transformation $U_{ij}\to\exp(i\gamma_i)U_{ij}$ where $\gamma_1=0$, $\gamma_2\simeq 0$, $\gamma_3\simeq\pi-\delta$. This corresponds to the identifications $\Phi=\phi_{23}$, $\phi_{12}=\phi_{13}=0$, $s_1=-s_{12}$, $s_3=s_{13}/s_{12}$, $\tan\delta=\sin\Phi[\cos\Phi-s_{13}/s_{12}s_{23}]^{-1}$, $s_2=s_{23}\sin\Phi/\sin\delta$. The definition of maximal *CP* nonconservation appropriate to this presentation is $\delta=\pm\pi/2$. This implies for the invariant phase Φ of (2), $\cos\Phi \simeq \pm|U_{ub}/U_{us}U_{cb}|$ rather than $\cos\Phi=0$. We thus see another illustration of the fact that maximal *CP* nonconservation in one presentation may lead to numerical values of physical *CP*-nonconserving quantities *different* from the ones obtained with maximal *CP* nonconservation in another scheme. Notice that the quantity $-\mathrm{Im\,Tr}\,U$ which measures the strength of all *CP* nonconservations can be rewritten using the identifications above as[10] $s_1^2s_2s_3\sin\delta$.

We point out that $-\mathrm{Im\,Tr}\,U$ becomes approximately $|U_{us}U_{ub}U_{cb}|$ when Φ is $\pi/2$ but $|U_{us}U_{ub}U_{cb}|\times[-1|U_{ub}/U_{cb}U_{us}|^2]^{1/2}$ when δ is $\pi/2$. Thus the numerical values of maximal *CP*-nonconserving amplitudes are larger in the presentations like (2) than in the KM-type presentations or any others. This might be considered a criterion for preferring (2), although maximality at the Lagrangian level can certainly be consistently defined in the KM scheme. It seems to us that the schemes like (2) are aesthetically preferable because they do not single out one generation from the others.

Finally, we remark that the magnitude of the *CP* impurity $\epsilon=(2\eta_{+-}-\eta_{00})/3$ is given, according to the standard recipe[11] with, however, the mixing convention based on Φ, by

$$|\epsilon|\simeq 3Bs_{13}s_{23}\sin\Phi\left[+0.4\ln\frac{m_t^2}{m_c^2}+0.6\frac{m_t^2}{m_c^2}s_{23}^2\left(1-\frac{s_{13}}{s_{12}s_{13}}\cos\Phi\right)\right]. \tag{10}$$

The main uncertainty in (10) is the parameter B which is expected to be in the range $\frac{1}{3}$ to 1. Provided that the top quark mass is less than 60 GeV, and using present data on s_{23} and s_{13}, (10) will be maximized for $\Phi=90°$. But notice that there is a rather slow dependence on Φ in this range. The variations from the maximum at $\Phi=90°$ out to 60° and 120° are at most 30% and 10%, respectively. Thus, as a practical matter, it would not be easy to conclude using our knowledge of ϵ that Φ is really maximal. In principle, the parameter $\epsilon'=(\eta_{+-}-\eta_{00})/3$ could shed some light on this question but the calculation[13] is hard to carry out to the required accuracy.

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(a)On leave from the Technion, Haifa, Israel.

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U_{11} and U_{33} real while $\omega_{13}\omega_{23}\omega_{12}$ has U_{22} and U_{33} real. For each case, a conjugate can be defined by multiplying the ω_{ij}'s in reverse order. This conjugate matrix has the same elements real along the diagonal as may be seen by noting that it may be gotten by taking the transpose of the original and setting $s_{ij} \rightarrow s_{ij}$ and $\phi_{ij} \rightarrow -\phi_{ij}$. The two conjugate matrices belong to different equivalence classes.

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