PHYSICAL REVIEW

LETTERS

VOLUME 54 | 4 FEBRUARY 1985 | NUMBER 5

Evaluation of Critical Exponents on the Basis of Stochastic Quantization

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(Received 17 September 1984)

In the context of stochastic quantization of field theories we propose a method to make analytic computations of critical exponents and we evaluate them for $(\boldsymbol{\phi}^2)_3^2$. It consists of regularization of the theory by use of a convenient non-Markovian process, where nonlocality in time is measured by the regularizing parameter σ. For a fixed dimensionality there is a value of σ where the theory is renormalizable and asymptotically free in the infrared, allowing a perturbative expansion around it.

PACS numbers: 03.70.+k, 05.70.Jk

Analytic computations of critical exponents for three-dimensional systems with a finite number of field components within the framework of perturbative field theory have been possible to do so far only by means of the ϵ-expansion method.[1] As is well known, the tree approximation yields mean-free values, but infrared (IR) divergences prevent us from computing loop corrections at any dimension less than four, in particular for $d=3$. A way out of this problem is to consider the case $d<4$ as an expansion around the four-dimensional theory; a second aspect of this expansion is that the β function has a zero of order $\epsilon = 4-d$ and the method also provides a small parameter to make a perturbative calculation.[2]

In this note we propose an analytical method to obtain exponents in perturbative field theory directly at the physical value of the dimensionality. In particular, we shall apply it to the interesting case of $(\boldsymbol{\phi}^2)^2$ in three dimensions.

The technique makes a nontrivial use of the stochastic quantization of field theory. This quantization procedure, introduced by Parisi and Wu,[3] has already been extensively used; see, e.g., Parisi *et al.*[4] It consists of introduction of an extra time dimension t and imposition of the following equation for the classical field:

$$(\partial/\partial t)\phi(x,t) = -[\delta/\delta\phi(x,t)]S + \xi(x,t), \qquad (1)$$

where S is the classical action and $\xi(x,t)$ is a Gaussian delta-correlated random force:

$$\langle \xi(x,t)\xi(x',t')\rangle = 2\delta^d(x-x')\delta(t-t'). \qquad (2)$$

In this formalism, Green's functions are obtained by the taking of the infinite-time limit of the average over ξ of the corresponding product of fields $\phi(x,t)$:

$$G(x_1,x_2,\ldots,x_n) = \lim_{t\to\infty} \langle \phi(x_1,t)\phi(x_2,t)\cdots\phi(x_n,t)\rangle. \qquad (3)$$

The field theory defined in this way reproduces the usual one; therefore, it still has the usual ultraviolet (uv) divergences and first it must be regularized. An interesting property of this quantization technique is that it also suggests new regularization procedures. The idea, introduced by Breit, Gupta, and Zaks,[5] is to replace the Markovian process defined in Ref. 2 by a non-Markovian one, i.e., to replace $\delta(t-t')$ by a function g_σ such that

$$\lim_{\sigma\to 0} g_\sigma(t-t') = \delta(t-t'). \tag{4}$$

A convenient form of g_σ is[6]

$$g_\sigma(t-t') = (\sigma/2)|t-t'|^{\sigma-1}, \tag{5}$$

which has the property that uv divergences appear as poles in σ.

At this point one can wonder whether there is a value of $\sigma=\sigma^*$ such that IR divergences can be handled and the corresponding β function has a zero $O(\rho)$ with $\rho=2(\sigma-\sigma^*)$. If such a system exists then the critical exponents of the original Markovian theory could be obtained by expansion of the non-Markovian ones. Following this plane we shall first find the value σ^* and then argue that the corresponding theory is renormalizable and asymptotically free in the IR. After that we shall compute the critical exponents to the lowest nonzero order in ρ.

Let us start by noticing that if we consider the interaction $S_{\rm int}=(\lambda_0/4!)\phi^4(x,t)$ and take $g_\sigma(t-t')$ as given in (5), then the coupling constant λ_0 has dimensions given by (μ having dimensions of mass)

$$[\lambda_0] = \mu^{2\sigma+\epsilon}, \tag{6}$$

which for $d=3$ gives a dimensionless coupling constant for $\sigma=-\frac{1}{2}$.

A power-counting analysis of the stochastic perturbative series indeed indicates that the theory is renormalizable for $\sigma=-\frac{1}{2}$. Let us recall that the usual perturbative expansion of field theory comes from stochastic quantization by solving first the Langevin equation (1) in terms of tree diagrams where one external line represents the field $\phi(t,x)$ and all the other external lines end on a stochastic source ξ, which graphically can be represented with a cross at the end. Every line corresponds to an integration over an intermediate time. A tree diagram, therefore, has one external line without cross $E_0=1$, a number I of internal lines $I=V-1$, V being the number of vertices, and a number E_c of crossed external lines easily seen to be $E_c=2V+1$. Green's functions are obtained from the average over ξ of products of ϕ's. This means contraction of the ξ's in pairs with use of Eq. (2), generating loops. We can now get diagrams with both internal and external lines with a cross coming from the contraction of two ξ's. To discuss renormalization, we focus on the one-particle irreducible parts of the Green's functions. From the integration over the intermediate times we can extract an overall integration over a time variable which we can call τ. This can be obtained for instance by use of polar coordinates in the multiple time integral. $\tau\to 0$ means that every intermediate time approaches t.

Given a diagram with L loops, m internal lines with a cross, and N intermediate times we easily obtain by inspection the overall integration:

$$\tau^{-(d/2)L}\tau^{m(\sigma-1)}\tau^{N-1}d\tau = \tau^{-D/2-1}d\tau. \tag{7}$$

Here the factor $\tau^{-(d/2)L}$ comes from the integration over momenta, $\tau^{m(\sigma-1)}$ comes from Eq. (5), and $\tau^{N-1}d\tau$ comes from the integration measure. The uv divergence comes from the behavior at $\tau\to 0$ and is expressed as a pole for $D=0$. We want to write the divergence degree D in terms of L and E_0, E_c. First, since every internal crossed line corresponds to two time variables, we have

$$N = 2m + V - 1 \tag{8}$$

and, moreover, for a Green's function with E_0, E_c external legs coming from a contraction of E_0 tree diagrams,

$$2m + E_c = 2V + E_0. \tag{9}$$

A standard counting in ϕ^4 theory gives the relation

$$V = L - 1 + (E_0 + E_c)/2, \tag{10}$$

which combined with Eq. (9) gives $m=L-1+E_0$; then using Eqs. (8) and (7) we obtain

$$D/2 = (d/2-2-\sigma)L + 3 + \sigma - (\tfrac{3}{2}+\sigma)E_0 - E_c/2. \tag{11}$$

The theory is renormalizable whenever D is independent of L, i.e., $\sigma^*=d/2-2$. In such a case we can reabsorb the divergences by redefining the parameters of the Langevin equation (1) which we, therefore, write in general as

$$Z_t(\partial/\partial t)\phi_R = (Z_\phi\Box - m_R^2 - \delta m^2)\phi_R + (\lambda_R/3!)\mu^{2\sigma+\epsilon}Z_V\phi_R^3 + Z_\xi\xi, \tag{12}$$

where we also introduced μ, the momentum scale of the renormalization procedure. For $d=4$ we obtain the usual field theory $\sigma^*=0$. For $d=3$, $\sigma^*=-\frac{1}{2}$. In this case, from Eq. (11) we get a logarithmic divergence for $E_0=1$, $E_c=3$ to be reabsorbed by Z_V. For $E_0=1$, $E_c=1$, we have a quadratic divergence reabsorbed by δm^2 (actually this will appear as m^2 times a pole at $D=0$) and two logarithmic ones defining Z_t and Z_ϕ.[7] Contrary to the $d=4$ case, here, there is no divergence for $E_0=2$, $E_c=0$ since $D=1$ does not correspond to a pole and therefore $Z_\xi=1$, as it is expected since the divergences appear as local terms in time and space in the Green's functions (i.e., they are proportional to δ functions on their derivatives). But for $\sigma\neq 0$ the correlation $g_\sigma(t-t')$ is not local in time and could not reabsorb divergences.

Finally, from Eq. (12) we compute the wave-function renormalization Z. Let us rescale the time $t=\alpha\bar{t}$ and the noise $\xi=\alpha^{(\sigma-1)/2}\bar{\xi}$ such that

$$\langle \bar{\xi}(t)\bar{\xi}(t')\rangle=\sigma|\bar{t}-\bar{t}'|^{\sigma-1}\delta^d(x-x').$$

With the choice of $\alpha=Z_t/Z_\phi$ this amounts to a field rescaling $\phi=Z^{1/2}\phi_R$, where $Z^{1/2}=Z_t^{-(\sigma-1)/2}\times Z_\phi^{(\sigma+1)/2}Z_\xi^{-1}$. For $d=3$ and $\sigma=-\frac{1}{2}$ we have

$$Z^{1/2}=Z_t^{3/4}Z_\phi^{1/4}. \qquad (13)$$

Of course, the usual coupling-constant renormalization constant Z_1 also involves Z_V times powers of Z_t and Z_ϕ. At the one-loop order, however, $Z_1=Z_V$.

We can define the infinite-time limit of Green's functions also for $\sigma\neq 0$. The existence of it being ensured by a mass gap,[8] we first take $t\to\infty$ and then approach the critical point. Furthermore, we can write for these Green's functions the standard renormalization-group equation. Since uv divergences appear as poles at $\rho=0$, we can calculate the renormalization constants by minimal subtraction of these poles. We find at the lowest order for $d=3$,

$$\begin{aligned} Z_1&=1+[\lambda_R 3/(2\sqrt{\pi})^3]/\rho,\\ Z_\phi&=1-[\lambda_R^2 R_\phi/2(4\pi)^3]/\rho,\\ Z_t&=1-[\lambda_R^2 R_t/2(4\pi)^3]/\rho, \end{aligned} \qquad (14)$$

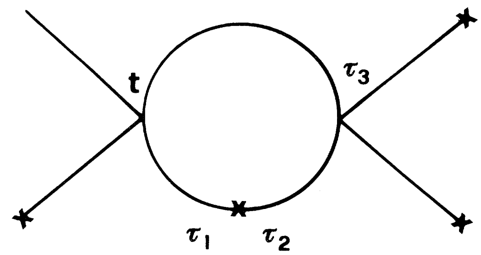

FIG. 1. The one-loop graph contributing to the β function.

where the renormalized coupling constant λ_R is given in terms of the bare one λ_0 by $\lambda_0=\lambda_R Z_1 Z^{-2}\mu^\rho$. The residues R_ϕ and R_t are given in terms of integrals over a finite hypercube in four dimensions, which have been computed numerically and give $R_\phi=0.196\pm 0.003$ and $R_t=0.264\pm 0.004$.

As an example, we show how to evaluate the one-loop diagram which contributes to the β function (Fig. 1). In computing its divergent part we can take zero momentum. As we take into account Eq. (5), it is given by (apart from a numerical factor)

$$I=\sigma\int\frac{d^3k}{(2\pi)^3}\int_0^t d\tau_1\int_0^t d\tau_3\int_0^{\tau_3}d\tau_2|\tau_1-\tau_2|^{\sigma-1}\exp\{-k^2[(t-\tau_3)+(\tau_3-\tau_2)+(t-\tau_1)]\}. \qquad (15)$$

To compute the integral over τ_1, τ_2, and τ_3 we divide it in all possible time orderings. Since t is the largest time and τ_3 is always greater than τ_2 we have only three regions: (a) $t\geq\tau_3\geq\tau_1\geq\tau_2$, (b) $t\geq\tau_3\geq\tau_2\geq\tau_1$, and (c) $t\geq\tau_1\geq\tau_3\geq\tau_2$, but actually the first two are equal for symmetry reasons. Considering case (a) we change variables: $\tau_1=t(1-\alpha_1)$, $\tau_2=t(1-\alpha_1\alpha_2)$, and $\tau_3=t(1-\alpha_1\alpha_2\alpha_3)$, obtaining after momentum integration

$$I_a=\sigma t^{\sigma+1/2}\frac{\pi^{3/2}}{(2\pi)^3}\int_0^1 d\alpha_1\,d\alpha_2\,\alpha_1^{\sigma-1/2}\alpha_2(1-\alpha_2)^{\sigma-1}(1+\alpha_2)^{-3/2}, \qquad (16)$$

and the integration over α_1 yields the pole at $\sigma=-\frac{1}{2}$. The last integral is also immediate but requires an analytical continuation in σ. After this is done we obtain, at the pole, a positive value for this integral: $I_a=[1/(2\sqrt{\pi})^3]/\rho$. I_c can be computed in a similar way, and after we consider the proper combinatorial factors we obtain Z_1 as in Eq. (14). The evaluation of Z_ϕ and Z_t is more lengthy but it involves the same difficulties.

We have also evaluated the renormalization constant Z_{ϕ^2} of the operator ϕ^2 which can be inserted into the Green's functions as a standard device to compute the dependence on the temperature of physical quantities near the critical point. We find at the lowest order

$$Z_{\phi^2}=1+[\lambda_R/(2\sqrt{\pi})^3]/\rho. \qquad (17)$$

From the renormalization constants we can compute the renormalization-group function and the anomalous dimensions. At the lowest order we obtain

$$\beta=\mu[\partial\lambda_R/\partial\mu]_{\lambda_0}=-\rho\lambda_R+\lambda_R^2 3(2\sqrt{\pi})^{-3}, \qquad (18)$$

given an IR stable fixed point at

$$\lambda_R^*=\rho(2\sqrt{\pi})^3/3 \qquad (19)$$

which vanishes for $\sigma=-\frac{1}{2}$, justifying in principle our perturbative computation as in the ϵ expansion. Of course, we then extrapolate to $\sigma=0$, i.e., $\rho=1$. The critical exponents γ and η are given in terms of anomalous dimensions at the fixed point. Defining $\gamma_{\phi^2}=\mu[\partial\ln Z_{\phi^2}/\partial\mu]_{\lambda_0}$ to lowest order, we have the critical exponent $\gamma=1-\frac{1}{2}\gamma_{\phi^2}$ and

$$\begin{aligned}\eta&=\mu[\partial\ln Z/\partial\mu]_{\lambda_0}\\ &=\tfrac{1}{2}\mu(\partial/\partial\mu)(\ln Z_\phi+3\ln Z_t). \end{aligned}\qquad (20)$$

From above we get $\gamma=1+\rho/6$ and $\eta=\rho^2(R_\phi+3R_t)/18$. The value of γ is the same as the one obtained from the ϵ expansion at the same order, i.e., $\gamma=1+\epsilon/6$, and in fact the factor 3^{-1} has in both cases the same combinatorial origin. At the lowest order in ρ we have $\gamma=1.167$ and $\eta=0.055$. On the other hand, the lowest order in ϵ gives $\eta=0.019$. The high-temperature series yields[2,9] $\gamma=1.250\pm0.003$ and $\eta=0.04\pm0.01$. We see that both the ϵ- and ρ-expansion values for η compare badly with the high-temperature value which lies somehow in between. Let us mention a more general point of view. The idea is to approach the physical point $\rho=\epsilon=1$ starting from an unphysical value in the line $\rho+\epsilon=1$, which according to Eqs. (6) and (11) corresponds to a renormalizable theory. For instance, the ϵ expansion chooses the direction of approach where $\rho=1$ and the ρ expansion the direction $\epsilon=1$. One could think also of an intermediate situation. Further work in this direction is currently being pursued.

Let us stress, however, that the series expansion in ρ is expected to be asymptotic, and the evaluation of more terms and the use of resummation techniques will be necessary as it occurs in the ϵ expansion.[9,10]

The above calculations can be easily extended to the $O(M)$ model, i.e., $S_{\text{int}}=\lambda_0/4!M(\phi^2)^2$. Evaluating the corresponding combinatorial factors we obtain

$$\gamma=1+\rho(M+2)/(2M+16),$$

and

$$\eta=\rho^2(R_\phi+3R_t)(3M+6)/2(M+8)^2,$$

i.e., M appears in the same way as in the ϵ expansion.

Two of the authors (J.A. and N.P.) would like to thank Professor Abdus Salam, the International Atomic Energy Agency, and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste, Italy.

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