Higher-Order Squeezing of a Quantum Field

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(Received 9 October 1984)

The concept of $2n$th-order squeezing of a quantum field is introduced as a natural generalization of the usual second-order squeezing. It is shown that the processes of degenerate parametric down conversion, harmonic generation, and resonance fluorescence all exhibit higher-order squeezing.

PACS numbers: 42.50.+q, 42.65.Bp

The subject of squeezing in quantized electromagnetic fields has recently received a great deal of attention,[1,2] perhaps because of the possibility of noise reduction in gravity-wave detection. A field in a squeezed quantum state exhibits fluctuations in one quadrature component $\hat{E}_1$ smaller than those in a coherent state, at the cost of increased fluctuations in the other quadrature component $\hat{E}_2$. If information could be impressed on and extracted from the $\hat{E}_1$ component, this could prove valuable in an optical communication channel. In principle, such a channel would be more efficient than one using completely coherent light, for which both quadrature components fluctuate equally.

In the past, attention was always focused on the quadratic deviation or dispersion $\langle(\Delta\hat{E}_1)^2\rangle$ as indicative of the field fluctuations. However, with the development of techniques for making higher-order correlation measurements in quantum optics, interest naturally extends to the higher moments of the field also. We therefore introduce a generalization of the squeezing concept, and examine some of its implications below.

For convenience we start with the usual definitions. Consider two quadrature components $\hat{E}_1,\hat{E}_2$ of one polarization of the electric field defined by[3]

$$\hat{E}_1=\hat{E}^{(+)}e^{-i\phi}+\hat{E}^{(-)}e^{i\phi},$$
$$\hat{E}_2=\hat{E}^{(+)}e^{-i(\phi+\pi/2)}+\hat{E}^{(-)}e^{i(\phi+\pi/2)}, \tag{1}$$

where $\hat{E}^{(+)}$ and $\hat{E}^{(-)}$ are the positive and negative frequency parts of the real field operator, and ϕ is a phase angle to be chosen. The commutator

$$[\hat{E}^{(+)},\hat{E}^{(-)}]=C \tag{2}$$

is a real positive c number, which is finite so long as the frequency decompositions of the fields in which we are interested do not extend to infinite frequencies. In some situations it is adequate to consider a single-mode field, but there are problems, like resonance fluorescence, where this is not appropriate. Then it follows that $\hat{E}_1,\hat{E}_2$ are canonical conjugates satisfying the commutation and uncertainty relations

$$[\hat{E}_1,\hat{E}_2]=2iC,$$
$$(\langle(\Delta\hat{E}_1)^2\rangle\langle(\Delta\hat{E}_2)^2\rangle)^{1/2}\geq C, \tag{3}$$

where $\Delta\hat{E}$ stands for the deviation $\hat{E}-\langle\hat{E}\rangle$. The state is described as squeezed if there is some phase angle ϕ for which

$$\langle(\Delta\hat{E}_1)^2\rangle<C, \tag{4}$$

and from Eq. (1) $\hat{E}_2$ can be regarded as a special case of $\hat{E}_1$. Also, with the help of the commutation relation (2), the normally ordered dispersion is given by

$$\langle:(\Delta\hat{E}_1)^2:\rangle=\langle(\Delta\hat{E}_1)^2\rangle-C, \tag{5}$$

and this evidently becomes negative for a squeezed state. Now $\langle:(\Delta\hat{E}_1)^2:\rangle$ vanishes for a coherent state of the field, and it is nonnegative for any state that is describable classically. It follows that the fluctuations of $\hat{E}_1$ in a squeezed state are smaller than in any coherent state, including the vacuum, and that such a state is purely quantum mechanical, without classical counterpart.

We now make a natural generalization of the foregoing, by calling the state squeezed to the $2N$th order in $\hat{E}_1$ ($N=1,2,3,\ldots,$), if there exists a phase angle ϕ, such that $\langle(\Delta\hat{E}_1)^{2N}\rangle$ is smaller than for a completely coherent state of the field. We have deliberately focused only on the moments of even order, because only in those cases is the state necessarily nonclassical. Once again it is useful to relate these higher-order moments to the normally ordered ones. By using the Campbell-Baker-Hausdorff identity in the form

$$\langle\exp(\Delta\hat{E}_1x)\rangle=\langle:\exp(\Delta\hat{E}_1x):\rangle\exp(\tfrac{1}{2}x^2C), \tag{6}$$

expanding both sides as a power series in x and comparing coefficients of x^r, we readily obtain the relation

$$\langle(\Delta\hat{E}_1)^N\rangle=\langle:(\Delta\hat{E}_1)^N:\rangle+\frac{N^{(2)}}{1!}(\tfrac{1}{2}C)\langle:(\Delta\hat{E}_1)^{N-2}:\rangle+\frac{N^{(4)}}{2!}(\tfrac{1}{2}C)^2\langle:(\Delta\hat{E}_1)^{N-4}:\rangle+\cdots$$
$$+\begin{cases}(N-1)!!C^{(1/2)N}, & \text{if } N \text{ is even},\\ \dfrac{N!}{3!2^{(1/2)N-3/2}}\dfrac{C^{(1/2)N-3/2}}{(\frac{1}{2}N-\frac{3}{2})!}\langle:(\Delta\hat{E}_1)^3:\rangle, & \text{if } N \text{ is odd}.\end{cases} \tag{7}$$

The normally ordered moments of the deviation all vanish for a coherent state, so that the field is squeezed to order $2n$ if

$$\langle (\Delta \hat{E}_1)^{2n} \rangle < (2n-1)!!C^n. \tag{8}$$

For example, we have squeezing to the fourth order if

$$\langle :(\Delta \hat{E}_1)^4: \rangle + 6C \langle :(\Delta \hat{E}_1)^2: \rangle < 0, \tag{9}$$

to the sixth order if

$$\langle :(\Delta \hat{E}_1)^6: \rangle + 15C \langle :(\Delta \hat{E}_1)^4: \rangle + 45C^2 \langle :(\Delta \hat{E}_1)^2: \rangle < 0, \tag{10}$$

etc. We now examine the question of whether higher-order squeezing exists in a number of systems that are already known to exhibit second-order squeezing.

Degenerate parametric down-conversion.— In this problem, which has been treated numerous times in the past,[4,5] a strong incident field of frequency 2ω interacts with a nonlinear crystal to generate light at the subharmonic frequency ω. We may treat the strong incident field as classical and of complex amplitude v, but the down-converted light of frequency ω has to be treated as quantized. We therefore write for the total energy

$$\hat{H} = \hbar \omega \hat{n} + hg[v e^{-2i\omega t} \hat{a}^{\dagger 2} + \text{H.c.}], \tag{11}$$

where $\hat{a}$, $\hat{a}^\dagger$ and $\hat{n}$ are annihilation, creation, and number operators for the down-converted mode, and g is a coupling constant. The general solution of the Heisenberg equation of motion has the form[5]

$$\hat{a}(t) = \cosh(2g|v|t) e^{-i\omega t} \hat{a}(0) - i(v/|v|) \sinh(2g|v|t) e^{-i\omega t} \hat{a}^\dagger(0). \tag{12}$$

As we are dealing with a single quantized mode, we may define the quadrature component $\hat{E}_1(t)$ by

$$\hat{E}_1(t) = \hat{a}(t) e^{-i\phi} + \hat{a}^\dagger(t) e^{i\phi}, \tag{13}$$

so that the commutator C in Eq. (2) is unity. Then if the initial quantum state is the vacuum, we find from Eqs. (12) and (13) that if the phase angle ϕ is chosen so that $2\omega t + 2\phi + \pi/2 - \arg v = \pi$, then

$$\langle (\hat{E}_1(t))^{2n} \rangle = (2n-1)!! e^{-4ng|v|t}. \tag{14}$$

Comparison with Eq. (8) shows that the parametrically down-converted field is squeezed to all even orders, not only to the second order. The state produced in this case is similar to the squeezed state that was introduced by Stoler,[6] which is a special case of the two-photon coherent state of Yuen,[7] which was also treated by Mollow.[5]

Harmonic generation.— The process in which an incident laser beam of the fundamental frequency ω interacts with a nonlinear medium to produce the harmonic at frequency 2ω is already known to squeeze the fundamental mode to the second order.[8] The Hamiltonian

$$\hat{H} = \hbar \omega \hat{n}_1 + 2\hbar \omega \hat{n}_2 + hg(\hat{a}_2^\dagger \hat{a}_1^2 + \text{H.c.}) \tag{15}$$

has the short-time solution for the fundamental mode $\hat{a}_1(t)$:

$$\hat{a}_1(t) e^{i\omega t} = \hat{a}_1(0) - 2igt\hat{a}_1^\dagger(0)\hat{a}_2(0) + 2g^2t^2[\hat{n}_2(0)\hat{a}_1(0) - \tfrac{1}{2}\hat{n}_1(0)\hat{a}_1(0)] + O(gt)^3. \tag{16}$$

If we define $\hat{E}_1$ for mode 1 as in Eq. (13), and take the initial state to be $|v\rangle_1 |\text{vac}\rangle_2$, in which $|v\rangle_1$ is the coherent state of complex amplitude v, we find to order $(gt|v|)^2$ that

$$\begin{aligned} \langle (\Delta \hat{E}_1)^2 \rangle &= 1 - 2g^2t^2|v|^2 \cos 2(\arg v - \phi), \\ \langle (\Delta \hat{E}_1)^4 &= 3 - 12g^2t^2|v|^2 \cos 2(\arg v - \phi). \end{aligned} \tag{17}$$

It follows that the choice $\arg v - \phi = n\pi$, which makes $\hat{E}_1$ squeezed to the second order, also ensures squeezing to the fourth order.

Resonance fluoresence from an atom.— We now consider the process of resonance fluorescence from a coherently excited two-level atom, which has been shown to yield second-order squeezing.[9] The positive frequency part of the electric field at a distant point $\mathbf{r}$ due to an atom at the origin that is being driven by a laser beam at frequency ω_1 has the general form[10,11]

$$\hat{E}^{(+)}(\mathbf{r},t) = K(\mathbf{r}) \hat{b}(t - r/c) + \hat{E}^{(+)}_{\text{free}}(\mathbf{r},t), \tag{18}$$

where $\hat{b}(t)$ is the atomic lowering operator and $K(\mathbf{r})$ is a geometric factor. If we are interested in normally ordered moments of the electric field at points $\mathbf{r}$ where the external field that is used to excite the atom vanishes, then $\hat{E}^{(+)}_{\text{free}}(\mathbf{r},t)$ operating to the right on the state, or its conjugate operating to the left, yields zero. With the use of the commutation relation[12]

$$[\hat{b}(t - r/c), \hat{E}^{(\pm)}_{\text{free}}(\mathbf{r},t)] = 0, \tag{19}$$

we can then evalute normally ordered moments of $\hat{E}_1$ given by Eq. (1) as if the free field operator in Eq. (18) were absent. With the help of the steady-state solution[11]

$$\langle \hat{b}(t) \rangle = \frac{-\frac{1}{2}\Omega/\beta(1+i\theta)}{\frac{1}{2}\Omega^2/\beta^2+1+\theta^2} \exp[i(-\omega_1 t+\alpha)], \quad (20)$$

where Ω is the Rabi frequency, 2β is the natural atomic linewidth, θ is the relative laser-atom detuning $(\omega_1-\omega_0)/\beta$, and α is the phase of the coherent field that excites the atom, we obtain

$$\langle \hat{E}_1(t) \rangle = \frac{|K|(\Omega/\beta)(1+\theta^2)^{1/2}}{\frac{1}{2}\Omega^2/\beta^2+1+\theta^2}\cos\psi,$$

with (21)

$$\psi = (-\omega_1 t+\alpha+\arg K-\phi+\tan^{-1}\theta+\pi).$$

Also, we find

$$\langle :\hat{E}_1^2: \rangle = \frac{\frac{1}{2}|K|^2\Omega^2/\beta^2}{\frac{1}{2}\Omega^2/\beta^2+1+\theta^2}, \quad (22)$$

while

$$\langle :\hat{E}_1^n: \rangle = 0 \quad \text{for } n > 2. \quad (23)$$

From these results we eventually obtain

$$\langle :(\Delta\hat{E}_1)^4: \rangle + 6C\langle :(\Delta\hat{E}_1)^2: \rangle = \frac{3|K|^2(\Omega/\beta)^2}{\frac{1}{2}\Omega^2/\beta^2+1+\theta^2}\left[\frac{|K|^2(\Omega/\beta)^2(1+\theta^2)\cos^2\psi}{(\frac{1}{2}\Omega^2/\beta^2+1+\theta^2)^2} \times \left(1-\frac{1+\theta^2}{\frac{1}{2}\Omega^2/\beta^2+1+\theta^2}\cos^2\psi\right) + C\left(1-\frac{2(1+\theta^2)\cos^2\psi}{\frac{1}{2}\Omega^2/\beta^2+1+\theta^2}\right)\right]. \quad (24)$$

Now it has been shown[9] that the choice $\psi = n\pi$ together with the condition $\frac{1}{2}\Omega^2/\beta^2 < 1+\theta^2$ ensures squeezing of the second order, although the amount by which $\langle(\Delta\hat{E}_1)^2\rangle$ is squeezed is always very small. If we make $\frac{1}{2}\Omega^2/\beta^2$ sufficiently small, then the first term on the right of Eq. (24) can be made as small as desired, while the second term is negative, so that squeezing of the fourth order is realized. Actually, a more careful consideration of the magnitudes of the two terms shows that the second term generally dominates over the first when $\frac{1}{2}\Omega^2/\beta^2 < 1+\theta^2$. Therefore, the condition for second-order squeezing also ensures fourth order—and even sixth order—squeezing, although the amount of the squeezing is always very small.

These examples all show that higher-order squeezing should be no more difficult to realize the second-order squeezing in some systems, although neither will be easy in practice. Indeed, a larger fractional reduction of the higher moments $\langle(\Delta\hat{E}_1)^{2n}\rangle$ $(n > 1)$ than of the second moment is achievable in principle, so that higher-order squeezing may be particularly interesting from the point of view of noise reduction.

This research was supported by the Office of Naval Research and by the National Science Foundation.

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