Quasiperiodic Patterns

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We present here a general framework to produce quasiperiodic tilings and more general quasiperiodic patterns in n dimensions corresponding to a finite number of local neighborings around each point. In particular, we give simple descriptions of the Penrose tilings of the plane and of a tiling of the three-dimensional space which exhibits an icosahedral symmetry. The Fourier transform of this last pattern is derived and shows a striking similarity with the electron-diffraction images obtained for a recently discovered alloy of Al and Mn.

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We present here a very general method to produce quasiperiodic tilings in arbitrary dimension. We generalize ideas already used by de Bruijn,[1] Mackay,[2] and Kramer and Neri[3] and we systematize the so-called projection method. The main idea is to generate such tilings of a p-dimensional space E as the projection, from a higher-dimensional space R^n, of a p-dimensional surface made up of a suitable union of p-facets of a regular n-dimensional lattice $\mathbf{L}$ in R^n.

Actually, let $\mathbf{L}=Z^n$ be the n-dimensional lattice in R^n generated by the natural basis $(e_1,...,e_n)$, and let γ_n be the unit cube. Let $E\subset R^n$ be a p-dimensional subspace of R^n, and assume that E does not contain any point of the lattice, except the origin.

There are $\binom{n}{p}$ different p-facets (the p-dimensional analog of an edge) of γ_n containing the origin. These facets project on E on *a priori* $\binom{n}{p}$ different p-volumes. A tiling of E by means of these volumes is obtained in the following way: Let $S=E+\gamma_n$ be the open strip generated by shifting γ_n along E; the claim is that the union of all p-facets entirely contained in S is exactly a p-dimensional surface of R^n, the projection of which on E gives the announced tiling.

If E' is the orthogonal complement of E in R^n, the projection K of the strip on E' is just the projection of the unit cube γ_n. Moreover, if $E\cap\mathbf{L}=(0)$, the projection of the lattice $\mathbf{L}$ in E' is one to one. Thus the set of vertices of the quasiperiodic tiling corresponds to the set of points of $\mathbf{L}$ that project in E' inside K.

Now, if E is invariant under the action of a subgroup G of the point group of the lattice $\mathbf{L}$ the *a priori* $\binom{n}{p}$ different tiles fall into classes, in such a way that the tiles of each class have the same shape, and are permuted by G.

The set of p-volumes around a given vertex x of the tiling is completely specified by the set of corresponding p-facets falling in K around the corresponding point x' in E'. Note that no $(p+1)$-, $(p+2)$-, . . . , n-facet of L can project in E' strictly inside K.

The two rhombs of the original Penrose tiling (see Refs. 1 and 2 and Penrose[4] and Gardner[5]), with angles $2\pi/5$, $3\pi/5$ for the thick one and $\pi/5$, $4\pi/5$ for the thin one, suggest application of this method in R^5. Let $\mathbf{L}=Z^5\subset R^5$ with the natural basis (e_1,e_2,e_3,e_4,e_5) and let $\mathbf{\Delta}$ be the principal diagonal of the lattice. $\mathbf{L}$ is invariant under the action of the group G of rotations around $\mathbf{\Delta}$, generated by the circular permutation g: $g(e_i)=e_{i+1}$. R^5 falls into three G-invariant subspaces: two two-planes $\mathbf{P}_1$ and $\mathbf{P}_2$ and of course $\mathbf{\Delta}$. Note that $\mathbf{P}_1\cap\mathbf{L}=(0)$.

The lattice $\mathbf{L}$ projects on $\mathbf{P}_1$ and $\mathbf{P}_2$ on dense Z-modules $\mathbf{L}_1$ and $\mathbf{L}_2$ generated as the linear combinations (with coefficients in Z) of the projections of $e_1, \ldots, e_5$ which point to the five vertices of a centered regular pentagon. The projection of $\mathbf{L}$ in $\mathbf{P}_2+\mathbf{\Delta}$ is contained in a set of equidistant planes parallel to $\mathbf{P}_2$. The ten two-facets of γ_5 project in $\mathbf{P}_1$ on the two Penrose rhombs, each one being repeated five times.

The strip is defined by $S=\mathbf{P}_1+\gamma_5=\{\xi_1+\zeta \mid \xi_1\in\mathbf{P}_1, \zeta\in\gamma_5\}$, where γ_5 is the open unit cube of R^5. S projects in $\mathbf{P}_2+\mathbf{\Delta}$ on a "rhombic icosahedron" with 20 faces and 22 vertices. More generally the strip can be translated in R^5: Let $S_t=\mathbf{P}_1+\gamma_5+t$ where t is any vector of $\mathbf{P}_2+\mathbf{\Delta}$. The projection in $\mathbf{P}_2+\mathbf{\Delta}$ is now $K_t=K+t$. The claim is that for any t, a nonperiodic tiling of P_1 is obtained by projecting $S_t\cap\mathbf{L}$ on $\mathbf{P}_1$: The points thus obtained are in one-to-one correspondence with those of $K_t\cap\mathbf{L}'$ in $\mathbf{P}_2+\mathbf{\Delta}$. Since no three-facet can project in the interior of K_t, it can be seen that the projection of $S_t\cap\mathbf{L}$ is the set of vertices of a tiling of $\mathbf{P}_1$ by means of the two Penrose tiles.

The original Penrose tiling is defined by means of "forcing rules" which rule out a certain number of local patterns around a point. Actually seven situations are allowed, up to rotations or inversion. In our framework, and for a given tiling, the set of rhombs around a point x_1 in $\mathbf{P}_1$ is completely specified by the set of two-facets around the corresponding point x' in $\mathbf{P}_2+\mathbf{\Delta}$ which fall inside K_t, and the set of rhombs around a point x_1 of the tiling in $\mathbf{P}_1$ is completely specified by the set of three-facets of K_t containing the corresponding point x' in $\mathbf{P}_2+\mathbf{\Delta}$. The partition of K_t given by intersections of all three-facets yields a decomposition into cells, each one corresponding to a specific type of pattern in $\mathbf{P}_1$. The actually realized patterns of the tiling are those for which the corresponding cells are intersected by $\mathbf{L}'$.

The original Penrose tilings are obtained for all S_t with $t \in \mathbf{P}_2$: The cells of K_t which intersect $\mathbf{L}'$ correspond to the seven possible situations. More general tilings are obtained for a generic translation t, and nine new situations can occur, including for instance the ten-pronged star. Actually, the set of nonisomorphic tilings can be labeled by $(\mathbf{P}_2 + \mathbf{\Delta})/\mathbf{L}'$, an uncountable set.

Notice that such a construction of the Penrose tilings and of their generalization does not involve any reference to their self-similarity properties, as in Refs. 4 and 5 and Levine and Steinhardt.[6]

The same general method can be worked out in view of the icosahedral symmetry. The simplest construction involves R^6 endowed with an orthogonal representation of the icosahedral group G, which permutes Z^6 and is such that R^6 falls into two G-invariant three-spaces, E and E', with nonequivalent irreducible representations of G.

The twenty different three-facets of $\mathbf{L} = Z^6$ fall in E (and E') on two different rhombohedra, with the same facets (with angles arctan2 and π−arctan2), each of which is repeated ten times. These are the rhombohedra considered by Mackay[2] and Kramer and Neri.[3] The open strip is defined by $S = E + \gamma_6$ where γ_6 is the open unit cube of R^6. It can be seen that the projections $\boldsymbol{\pi}(\mathbf{L})$ on E and $\boldsymbol{\pi}'(\mathbf{L})$ on E' are Z modules which are dense in E and E'. If $e_1, \ldots, e_6$ is the natural basis of R^6, $\boldsymbol{\pi}(\pm e_1), \ldots, \boldsymbol{\pi}(\pm e_6)$ point to the twelve vertices of a regular icosahedron centered at the origin of E and so do $\boldsymbol{\pi}'(\pm e_1), \ldots, \boldsymbol{\pi}'(\pm e_6)$ in E'.

The projection $\boldsymbol{\pi}(S \cap \mathbf{L})$ in E is in one-to-one correspondence with the other projection $\boldsymbol{\pi}'(S \cap \mathbf{L}) = K \cap \mathbf{L}'$ where $K = \boldsymbol{\pi}'(\gamma_6)$ is a rhombic triacontahedron. The same arguments insure that no four-facet of Z^6 can project inside K and that a nonperiodic tiling of E by means of two different rhombohedra is obtained. The local pattern around a given point x in E is completely specified by the set of three-facets around the corresponding point x' in E'. As in the Penrose case, a cell decomposition is obtained which yields 24 possible patterns of rhombohedra around a point. In particular, the central cell of the triacontahedron corresponds to a twenty-pronged star with the icosahedral symmetry.

We present in Figs. 1, 2, and 3 three sections of this tiling, which are associated to axes of order 5, 3, and 2, respectively. The cuts, which are made along two-dimensional surfaces taken from two-facets of the tiling, yield quasiperiodic tilings of the plane. Actually, the first section, carried out orthogonally to a fivefold axis, projects on a generalized Penrose tiling. The two other sections project on quasiperiodic tilings of the plane associated with threefold and twofold symmetries. As in the two-dimensional case, the strip can be translated, which yields an uncountable set of nonisomorphic tilings.

One of the most striking features of all these tilings is their quasiperiodicity, in the sense that for any tiling, the measure defined by a Dirac delta at each vertex is quasiperiodic: Its Fourier transform is a sum of weighted Dirac measures with support in the Z module generated by the projections of the basis vectors of the lattice. For instance, in the icosahedral case, the

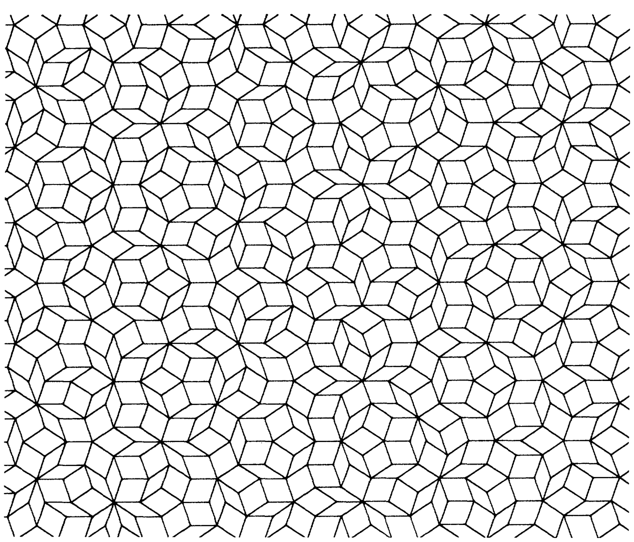

FIG. 1. A section of the three-dimensional tiling orthogonal to a fivefold axis: a generalized Penrose tiling.

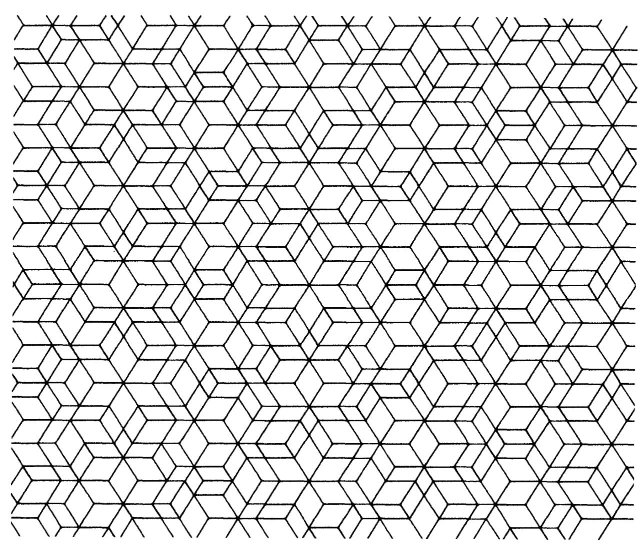

FIG. 2. A section of the three-dimensional tiling orthogonal to a threefold axis.

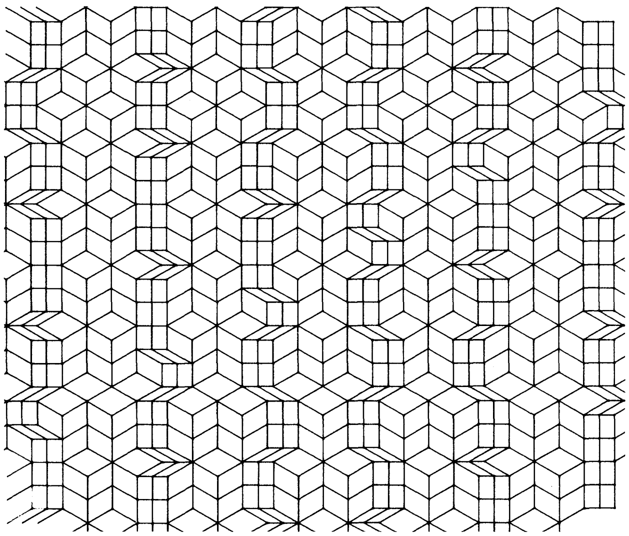

FIG. 3. A section of the three-dimensional tiling orthogonal to a twofold axis.

Fourier transform is the restriction of a six-dimensional Fourier transform which can be easily obtained as the convolution of the dual lattice of Z^6 by the Fourier transform of the characteristic function of the strip. Moreover, all tilings have the same Fourier transform, up to a phase at each point, in such a way that they are identical from a "diffractional" point of view.

Let $\nu=\chi_S\Sigma_{\xi\in Z^6}\delta_\xi$ be the measure associated to $S\cap Z^6$, where χ_S is the characteristic function of the strip in R^6. Then $n=\Sigma_{\xi\in Z^6}\chi_S(\xi)\delta_{\pi(\xi)}$ is the measure associated to the tiling in E. If $\xi=(x,x')$ is the orthogonal decomposition of $\xi\in R^6$ in E and E', and if $\kappa=(k,k')$ is the corresponding decomposition in the dual space, the Fourier transform of n is given by $\tilde{n}(k)=\nu(k,0)$. Now $\nu=\chi_S*\Sigma_{\lambda\in Z^6}\delta_\lambda$, and since $\chi_S(x,x')=\chi_{TR}(x')$ where χ_{TR} is the characteristic function of the triacontahedron in E', $\chi_S(k,k')=\delta(k)\chi_{TR}(k')$. Finally, $\tilde{n}(k)=\Sigma_{\lambda\in Z^6}\delta(k-l)\times\chi_{TR}(-l')$ where $\lambda=(l,l')$ is the decomposition of λ in the dual space.

We give in Figs. 4, 5, and 6 the computed Fourier transforms of a tiling, in the three planes respectively orthogonal to symmetry axes of order 5, 3, and 2. These patterns are astonishingly similar to the electron diffraction images obtained by Shechtman *et al.*[7] from rapidly cooled alloys of Al and Mn.

The general framework presented in this Letter can be generalized in many ways. For instance, if the condition which insures an exact tiling of the space is removed, more general quasiperiodic patterns are obtained. On the other hand, the projection space and the strip may have different orientations: The first one specifies the tiles while the second one dictates their relative abundance. As a matter of fact, the quasiperiodic tilings thus obtained can be seen to interpolate between periodic ones, which correspond to rational orientations of the strip.

Notice that the quasiperiodicity of the tilings is in-

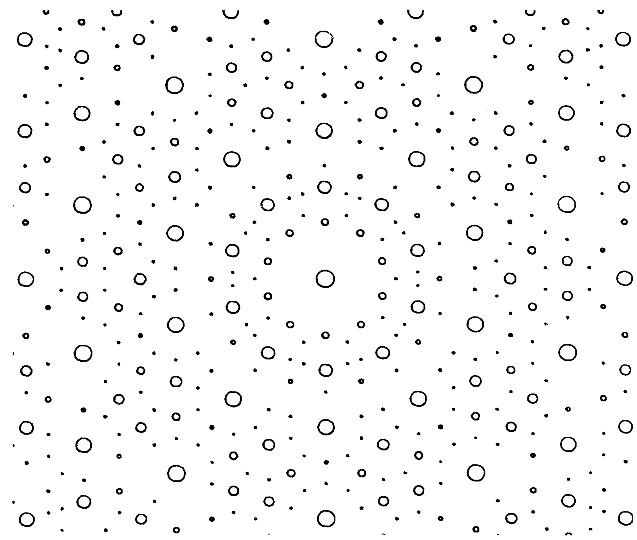

FIG. 4. Fourier transform in a plane orthogonal to a fivefold axis. In this figure and in the following ones, the circles are centered at the location of the Dirac measures and the radii are proportional to their amplitudes above a given threshold.

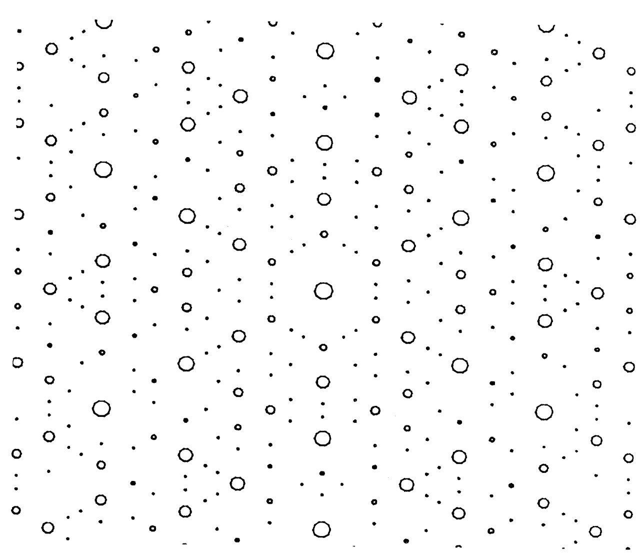

FIG. 5. Fourier transform in a plane orthogonal to a threefold axis.

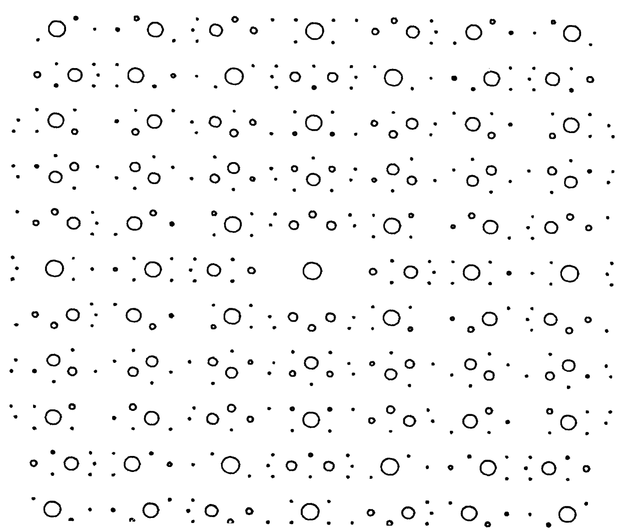

FIG. 6. Fourier transform in a plane orthogonal to a twofold axis.

dependent of the fact that neither the pentagonal nor the icosahedral symmetry is compatible with periodic ordering. In fact, the sections corresponding to the twofold and threefold axes (given in Figs. 2 and 3) show quasiperiodic tilings of the plane while these symmetries are of crystal type.

As a final remark, let us stress the idea that the construction presented herein works in any number of dimensions and with any lattice. For instance, it is easy to see that another type of quasiperiodic tiling of the space with icosahedral symmetry can be obtained from R^{10}, the icosahedron being replaced by a dodecahedron.

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Note added.—A recent preprint of V. Elser, kindly transmitted to us by D. Gratias, presents essentially the same results as ours.

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