Superstrings at High Temperature

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The behavior of ten-dimensional superstrings at high temperatures is studied by use of the microcanonical ensemble. We focus on closed superstring theories. The massive string excitations alone have negative specific heat. Such a system can reach equilibrium with the zero modes of the string if the energy carried by zero modes is less than a given fraction of the total energy. The temperature can then exceed the Hagedorn temperature. Some consequences for the evolution of the universe are discussed.

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The recent construction of anomaly-free superstring theories in ten dimensions[1,2] has drawn wide interest. These theories have phenomenologically realistic low-energy limits and they are believed to be ultraviolet finite. Furthermore, the familiar gravitational and Yang-Mills interactions are contained in the low-energy expansion of these theories. This encourages the belief that a consistent quantum theory that unifies all known interactions may be at hand. Some elegant compactification schemes which yield a low-energy supergravity theory coupled to a supersymmetric grand unified theory have also been proposed.[3] If the fundamental interactions of matter are really described by a superstring theory, then the evolution of the Universe may differ from that of the standard cosmological model, especially at scales approaching the Planck mass.

The first step in formulating a string cosmology is to understand the behavior of strings at high energy density and temperature. In this paper we study the thermodynamics of ten-dimensional superstrings with particular emphasis on the more phenomenologically promising heterotic string.[2] We shall see that open superstring models have strikingly different thermodynamic properties from closed superstring models. While the canonical ensemble provides a reasonable thermodynamic description of a gas of open superstrings, it fails to describe the statistical mechanics of a gas of closed superstrings. It is essential to use the microcanonical ensemble to understand the behavior of this system. This is highlighted by the fact that the massive excitations of a closed superstring system have negative microcanonical specific heat. In this respect their behavior has many similarities to that of a black hole.

To calculate the thermodynamic observables of a string gas, we must derive the density of states. Heterotic superstrings[2] are described by transverse and fermionic coordinates

$$\begin{aligned} x^i(\tau-\sigma) &= \tfrac{1}{2}x^i + \tfrac{1}{2}p^i(\tau-\sigma) + \frac{i}{2}\sum_{n\neq 0}\frac{\alpha_n^i}{n}e^{-2in(\tau-\sigma)}, \\ x^i(\tau+\sigma) &= \tfrac{1}{2}x^i + \tfrac{1}{2}p^i(\tau+\sigma) + \frac{i}{2}\sum_{n\neq 0}\frac{\tilde{\alpha}_n^i}{n}e^{-2in(\tau+\sigma)}, \\ S^a(\tau-\sigma) &= \sum_{n=-\infty}^{+\infty} S_n^a e^{-2in(\tau-\sigma)}, \end{aligned} \tag{1}$$

where

$$\begin{aligned} &[x^i,p^j] = i\delta^{ij}, \\ &[\alpha_n^i,\alpha_m^j] = [\tilde{\alpha}_n^i,\tilde{\alpha}_m^j] = n\delta_{n+m,0}\delta^{ij}, \\ &[\alpha_n^i,\tilde{\alpha}_m^j] = 0, \quad \{S_n^a,\bar{S}_m^b\} = (\gamma^+ h)^{ab}\delta_{n+m,0}, \end{aligned} \tag{2}$$

with $h=\frac{1}{2}(1+\gamma_{11})$, S^a a ten-dimensional Majorana-Weyl light cone spinor, and $0<\sigma<\pi$, $i,a=1,\ldots,8$. In this basis the mass operator M is given by $\alpha^1(M)^2 = 4N$, with N an eigenvalue of the operator

$$N = \sum_{n=1}^{\infty}(\alpha_{-n}^i\alpha_n^i + \tfrac{1}{2}n\bar{S}_{-n}\gamma^- S_n). \tag{3}$$

To determine the degeneracy of the eigenvalues of N, we use the generating function $P(x)=\sum_{N=1}^{\infty}P(N)x^N$, where $P(N)$ is the degeneracy of eigenvalue N. For the Veneziano 26-dimensional bosonic string $P_B(x) = \{\prod_{n=1}^{\infty}(1-x^n)\}^{-24}$. The ten-dimensional superstring is the product of bosonic and fermionic generating functions,

$$P_F(x) = \left(\frac{\prod_{n=1}^{\infty}(1+x^n)}{\prod_{n=1}^{\infty}(1-x^n)}\right)^8 = [\theta_4(0,x)]^{-8}, \tag{4}$$

where θ_4 is the Jacobi θ function.[4,5] The heterotic string is a hybrid of the bosonic and fermionic strings.

The degeneracy of states with mass M is given by[2]

$$d(M)=\text{const}\times P_F(\alpha^1 M^2/4)P_B'(\alpha^1 M^2/4),$$

where P_B' is slightly different from P_B because of the contribution of states from the compactification on the sixteen-dimensional self-dual even integral lattice,[2]

$$P_B'(x)=\left(\prod(1-x^n)\right)^{-24}\left[1+\sum_{m=2,4,\ldots}480\sigma_7(m)x^{2m}\right],$$

where $\sigma_7(m)$ is the sum of the seventh powers of the divisors of m. The behavior of $P(N)$ for large N has been determined by Huang and Weinberg,[6] generalizing the work of Hardy and Ramanujan.[7] It is given by $P_B^D(N)\sim N^{-(D+3)/4}\exp[2\pi(D/6)^{1/2}\sqrt{N}]$ for D transverse dimensions. For $D=24$, it is thus $P_B^{24}(N)\sim N^{-27/4}\exp(4\pi\sqrt{N})$. To determine $P(N)$, we use contour integration,

$$P_F(N)=(1/2\pi i)\int_c dz/[z^{n+1}\theta_4^8(0,z)].$$

We then map $\theta_4(0,z)$ to $\theta_2(0,e^{\pi^2/\ln z})$, since $e^{\pi^2/\ln z}$ is small for z approaching the stationary point $z\to 1^-$. Since $\theta_2(0,q)\sim 2q^{1/4}$ for small q, this gives

$$\theta_4(0,z)=(-\pi/\ln z)^{1/2}\theta_2(0,e^{\pi^2/\ln z})\underset{z\to 1^-}{\longrightarrow}2(-\pi/\ln z)^{1/2}e^{\pi^2/4\ln z}.$$

We then evaluate the contour integral by the saddle-point method which gives $P_F(N)\sim N^{-11/4}\exp(\pi\sqrt{8}\sqrt{N})$. To determine P_B' we need the sum $\sum_{m=0}^{N/4}P_B(N-4m)\sigma_7(2m)$. Using $\sigma_7(m)\sim m^7$, we find $P_B'(N)\sim N^{-11/4}\times\exp(4\pi\sqrt{N})$. The density of states in mass space is then

$$\rho(m)\sim m\,d(m)\sim[(\alpha^1)^{1/2}m]^{-10}\exp[(2+\sqrt{2})\pi(\alpha^1)^{1/2}m]$$

and is of the generic form

$$\rho(m)=cm^{-a}\exp(bm),\tag{5}$$

with $a=10$ and $b=(2+\sqrt{2})\pi(\alpha^1)^{1/2}$ for the heterotic string. The corresponding analysis for open (type I) superstrings yields $a=\frac{9}{2}$ and $b=\pi\sqrt{8}(\alpha^1)^{1/2}$, while for conventional closed superstrings (with no Yang-Mills gauge group) one finds $a=10$ and $b=\pi\sqrt{8}(\alpha^1)^{1/2}$.

One approach for discussion of the thermodynamics of such strings is the canonical ensemble. The partition function is

$$\ln Z=\frac{V}{(2\pi)^9}\int\rho(m)\,dm\int d^9k\ln\left(\frac{1+\exp[-\beta(k^2+m^2)^{1/2}]}{1-\exp[-\beta(k^2+m^2)^{1/2}]}\right),\tag{6}$$

where V is the nine-dimensional volume. Expanding the logarithm and integrating over the momentum gives

$$\ln Z\sim V\sum_{n=0}^{\infty}\left(\frac{1}{2n+1}\right)^5\int_\eta^\infty dm\,m^{-a+5}K_5[(2n+1)\beta m]\,e^{bm},\tag{7}$$

where η is the infrared cutoff below which the asymptotic form of the density of states Eq. (5) is no longer valid and $K_n(x)$ is the modified Bessel function. Using the asymptotic form of $K_n(x)$, $K_n(x)\underset{x\to\infty}{\longrightarrow}(1/\sqrt{x})\,e^{-x}$, we see that

$$\ln Z\sim\left(\frac{TT_0}{T_0-T}\right)^{-a+11/2}\Gamma\left(-a+\frac{11}{2},\eta\left(\frac{T_0-T}{TT_0}\right)\right),$$

where $\Gamma(a,x)$ is the incomplete gamma function and $T_0=1/b$. The canonical partition function diverges for $T>T_0$ and thus T_0 seems to describe a maximum temperature for thermodynamic equilibrium.[8] The thermodynamic observables of interest may then be calculated from Z: $P=T\partial\ln Z/\partial V$, $C_V=d\langle E\rangle/dT$, and $\langle E\rangle=T^2\partial\ln Z/\partial T$.

For $a\leq\frac{13}{2}$, the pressure, energy density, and specific heat diverge as the temperature approaches T_0. This behavior is consistent with the idea that T_0 is the maximum temperature of the system. For $a>\frac{13}{2}$, however, we see that the pressure and energy density are constant as the temperature approaches T_0, and for $a>\frac{15}{2}$, the specific heat is also constant as T approaches T_0. This behavior is contrary to the notion that T_0 is the limiting temperature of the system. Since the energy density or specific heat does not diverge at T_0, there is nothing to prevent one from passing through T_0 by pumping energy into the system, and yet the canonical ensemble does not provide a description of the system for this range of observables.

This peculiar behavior warrants careful examination. In particular it is wise to check the size of energy fluctuations. We find that the mean-square energy fluctuation $[\langle E^2\rangle-\langle E\rangle^2]/\langle E\rangle^2$ is larger than 1 for $\rho=E/V>\rho_0=(1/a-\frac{13}{2})(\eta^{13/2-a})(2\pi b)^{-9/2}$.

When the energy fluctuations are of this magnitude, the canonical ensemble is no longer a good thermodynamic description of the system, and one should reexamine the system using the more fundamental microcanonical ensemble. In this ensemble the total energy E is fixed and one counts the number of microstates which yield a given macrostate,

$$\Omega(E,V)=\sum_{n=1}^{\infty}\left[\frac{V}{(2\pi)^9}\right]^n\frac{1}{n!}\prod_{i=1}^{n}\int_{\eta}^{\infty}\rho(m_i)\,dm_i\int d^9p_i\,\delta(\textstyle\sum E_i-E)\,\delta(\textstyle\sum p_i). \tag{8}$$

Following Frautschi,[9] and Carlitz,[10] we find that for $a>\frac{13}{2}$ and $\rho>\rho_0$, the density of states as a function of energy has the same functional form as the density of states as a function of mass, $\Omega(E,V)\simeq VE^{-a}\times\exp(bE)$.

This density of states thus applies to the description of heterotic strings but not to that of open superstrings. This is the fundamental source of the difference in their thermodynamic behavior. Furthermore, the most probable number (n) of strings is such that the favored configuration is for $n-1$ strings to carry as little energy as possible and for one string to carry the remaining energy.[9,10] The condition $\rho>\rho_0$ is equivalent to the condition $E\gg n\eta$. Therefore, $n-1$ strings like to carry energy η and the remaining string the energy $E-(n-1)\eta$, which is much greater than η. This is the source of the large energy fluctuations in the canonical ensemble. The favored thermodynamic configuration is highly inhomogeneous.

There is another remarkable fact which indicates the failure of the canonical ensemble to describe the physics of a gas of hot heterotic strings. Given the microcanonical density of states $\Omega(E,V)$, we can compute the microcanonical thermodynamic observables. The entropy S is $S=\ln\Omega(E,V)=-a\ln E+bE$. The temperature T is formally given by $T=(\partial s/\partial E)^{-1}=E/(bE-a)$. For positive energy E_s, the temperature exceeds $1/b$. The specific heat is $C=(-1/T^2)(\partial^2S/\partial E^2)^{-1}=(-1/T^2)(E^2/a)$. This is negative! In the canonical ensemble the specific heat is proportional to the mean-square energy fluctuations and is thus always positive.[11,12] A gas consisting purely of massive superstring excitations behaves in many respects like a black hole.[13] Recall that a black hole of mass M is characterized by a temperature $T=1/(8\pi M)$, where we have set G equal to 1. The specific heat dM/dt is also negative. A black hole (and the massive string excitations) can never be in thermal equilibrium with an infinite heat reservoir. Imagine a body of negative specific heat immersed in an infinite heat bath of temperature T. Suppose now that the temperature of the body fluctuates above that of the reservoir. Heat will then flow from the body to the reservoir. This lowers the energy of the body and raises its temperature. The body thus radiates even more heat until it evaporates completely. This is clearly an unstable situation. A body with negative specific heat can, in contrast, be in equilibrium with a finite-heat bath. In the black-hole case the condition for stable equilibrium is that the energy of the heat bath (in this case radiation) be less than one quarter the mass of the black hole.

We now derive the corresponding condition for the massive excitations of a heterotic string to be in equilibrium with its massless modes (radiation). Let the total energy of the massive excitations be E_s and the energy of the massless excitations be E_r. The total number of configurations for this system is $\exp(S_r+S_s)$, where S_s,S_r denote the entropy of the massive and massless excitations, respectively. The most probable values of E_s and E_r will be those which maximize S_s+S_r with the constraint that the total energy is fixed. This means $\partial S_s/\partial E_s=\partial S_r/\partial E_r$ and $\partial^2S_s/\partial E_s^2+\partial^2S_r/\partial E_r^2<0$. The first condition implies $T_r=T_s=T$ and the second may be rewritten as

$$(1/T)(\partial T/\partial E_r)>T(\partial^2S_s/\partial E_s^2)=aT/E_s^2=(1-bT)^2/aT.$$

Now in ten-dimensional space-time the energy of a massless gas of bosons and fermions is given by $E=\sigma VT^{10}$ with $\sigma=(8\pi^5/3465)[n_b+(1-1/2^9)n_f]$. The above inequality becomes an equality for a (maximum) temperature T_c given by

$$bT_c=\frac{20bE-9a\pm(81a^2+40abE)^{1/2}}{20(bE-a)}.$$

Given this maximum temperature, we know that $E_r<(E_r)_{\max}=E_{\text{total}}+aT_c/(1-bT_c)$. The condition for equilibrium between the massive and massless excitations is then $V<(E_r)_{\max}/\sigma T_c^{10}$. As an example, consider $E=100M_P$, where M_P is the Planck mass. We choose the string tension such that $b=1/M_P$. Then $T=1.18M_P$, $(E_r)_{\max}=36M_P$, and V is less than $1.2\times10^{-4}(M_P)^{-9}=(0.37/M_P)^9$. We have used $\sigma=5.7\times10^4$ corresponding to $n_b=n_f=4032$ for the massless modes of the heterotic string. For the total energy E very large, these results become $T_c\cong M_P$, $(E_r)_{\max}\simeq E$, and $V<b^{10}E/\sigma$.

If the volume of the system is below this critical value for a given total energy E, then the massive string excitations can exist in equilibrium with the massless excitations at a temperature between T_0 and T_c. When the volume is greater than this critical value, the massive string excitations cannot be in equilibrium with radiation at any temperature. They must decay into massless modes to achieve equilibrium. One important offshoot of this discussion is the realization that a system consisting of heterotic strings can reach equilibrium above the Hagedorn temperature, provided that the energy in the higher string modes is greater than a certain fraction of the total energy. There is no maximum energy density either.

The microcanonical ensemble description of a gas of type-I (open) superstrings is more complicated to discuss. If the energy is below a critical value, the microcanonical predictions are equivalent to the canonical predictions. When the energy exceeds this value, the energy fluctuations are large and the microcanonical ensemble again becomes the fundamental ensemble. The density of states can be computed[10] and has the same basic features as the one discussed. It is, however, more complicated. Details will be presented.[14]

We conclude by discussing the possible evolution of the universe for a heterotic string theory. This is very sensitive to the initial conditions. We may instead imagine the time-reversed process of the present universe contracting. As it contracts, its temperature would rise and the universe would consist of a gas of massless string excitations in thermal equilibrium. As the temperature approaches T_0, the higher string modes can be excited. When the volume of the universe falls below the critical value, there is a phase transition to a system of massive and massless string excitations in thermal equilibrium above the temperature T_0. Of course, the dynamics of string interactions and the compactification of the ten-dimensional space to four dimensions may alter our conclusions. These questions are presently under study.

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