High-Order Strong-Coupling Calculation of the Ground-State Energy Density in Supersymmetric Field Theory

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This paper is a continuation of an earlier study of the ground-state energy density E_0 in two-dimensional supersymmetric field theories. Lattice strong-coupling techniques are used to calculate E_0 to eleventh order, extending the previous calculation by four additional orders. Our results continue to suggest that supersymmetry invariance is restored in the continuum limit. Moreover, new and unexpected large-order behavior in the lattice perturbation-expansion coefficients is revealed in this high order of perturbation theory.

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In a previous Letter[1] we reported a strong-coupling lattice calculation of the ground-state energy density E_0 for a class of two-dimensional supersymmetric Wess-Zumino field theories whose Euclidean Lagrangian is given by

$$L_E = \tfrac{1}{2}(\nabla_\mu\phi)^2 + \tfrac{1}{2}\bar{\psi}\gamma_\mu\nabla_\mu\psi + \tfrac{1}{2}gS'(\phi)\bar{\psi}\psi + \tfrac{1}{2}g^2[S(\phi)]^2. \quad (1)$$

Here, ψ is a two-component Majorana field, ϕ is a real scalar field, and

$$S(\phi) = \phi^{2k+1}, \quad k = 0, 1, 2, \ldots . \quad (2)$$

It is known that[2,3] supersymmetry is not broken dynamically for this class of theories, and that therefore the ground-state energy density must vanish. Our motivation for performing a direct calculation of E_0 was to see if lattice strong-coupling calculations are useful for theories having a supersymmetry invariance: Introducing a lattice scale a explicitly breaks supersymmetry.[4] However, if in the continuum limit we find that $E_0 = 0$, then we have a signal that supersymmetry is restored as $a \to 0$. On the basis of a seventh-order calculation of E_0, we concluded in Ref. 1 that a lattice regularization does not break supersymmetry permanently.

We have now extended the strong-coupling calculation of E_0 to eleventh order. We report the results of this calculation in this Letter. In brief, our findings continue to indicate that supersymmetry is restored in the continuum limit $a \to 0$.

It is known that the conventional lattice extrapolation methods used in this paper and in Ref. 1 fail in other quantum theories in sufficiently high order.[5] For example, for the anharmonic oscillator whose Hamiltonian is given by

$$H = p^2/2 + gx^{2N}, \quad (3)$$

optimal accuracy in the strong-coupling expansion is achieved in order five for $N = 2$; starting with order six, the accuracy gradually decreases with increasing order of perturbation theory. Thus, the lattice extrapolants appear to behave like the partial sums of a divergent asymptotic series; they approach the exact solution for a while and then veer away. Furthermore, the order of perturbation theory in which optimal accuracy is achieved gradually decreases as N increases. For the supersymmetric theory of Eq. (1) with $k = 1$, the accuracy appears to worsen after order seven. The order-eight result for E_0 is slightly less accurate (it appears to diverge away from 0) and in order nine the accuracy decreases slightly further. However, a surprising turnabout occurs in still higher orders of perturbation theory. The order-ten value represents a dramatic *improvement* in accuracy and in order eleven there is a further improvement. It is possible that this faltering in orders eight and nine is a temporary effect and that this series converges. At least it has not yet passed its optimum at eleventh order.

The procedure for calculating E_0 is completely described in Ref. 1. The strong-coupling lattice expansion for E_0 has the form

$$E_0 = g^2z^2\sum_{n=0}^{\infty} z^n A_{n+1}, \quad (4)$$

where $z = (ag)^{-2/3}$. The coefficient A_n is computed

TABLE I. Numerical values of the first 22 strong-coupling vertices for the $k=1(\phi^6)$ supersymmetric theory. These vertices are accurate to sixteen decimal places. This accuracy is required to calculate A_n dependably for large n because there is extensive cancellation between positive and negative graphs.

n	V_n
1	$0.830\,860\,925\,029\,559\,1\epsilon a$
2	$0.802\,380\,574\,875\,330\,7\epsilon^2 a^2$
3	$-0.333\,333\,333\,333\,333\,3\epsilon^3 a^5$
4	$-1.100\,582\,835\,782\,239\epsilon^4 a^6$
5	$1.181\,891\,464\,634\,797\epsilon^5 a^9$
6	$6.497\,529\,551\,395\,417\epsilon^6 a^{10}$
7	$-10.624\,503\,950\,713\,97\epsilon^7 a^{13}$
8	$-81.757\,589\,941\,742\,91\epsilon^8 a^{14}$
9	$178.513\,511\,520\,879\,7\epsilon^9 a^{17}$
10	$1766.139\,217\,438\,574\epsilon^{10} a^{18}$
11	$-4821.140\,637\,900\,176\epsilon^{11} a^{21}$
12	$-58\,297.858\,421\,057\,72\epsilon^{12} a^{22}$
13	$190\,970.838\,808\,033\,5\epsilon^{13} a^{25}$
14	$2\,729\,105.834\,435\,479\epsilon^{14} a^{26}$
15	$-10\,429\,959.094\,478\,07\epsilon^{15} a^{29}$
16	$-171\,982\,325.377\,497\,9\epsilon^{16} a^{30}$
17	$751\,169\,779.772\,133\,2\epsilon^{17} a^{33}$
18	$14\,037\,733\,051.973\,69\epsilon^{18} a^{34}$
19	$-68\,976\,921\,338.065\,72\epsilon^{19} a^{37}$
20	$-1\,440\,679\,437\,693.430\epsilon^{20} a^{38}$
21	$7\,865\,596\,742\,102.632\epsilon^{21} a^{41}$
22	$181\,577\,035\,737\,305.2\epsilon^{22} a^{42}$

from the set of all vacuum graphs having $2l$ fermion lines and $n-l$ boson lines, where $l=0,1,\ldots,n$. A_n is computed by multiplying the lattice sum of each graph (which we compute in coordinate space by summing over the positions of all vertices) with the appropriate symmetry number and the values of the vertices, summing over all graphs, and multiplying the result by -1. It is nontrivial to compute the lattice sum of nonplanar graphs which arise in large orders of perturbation theory. Analytical techniques for evaluating such graphs are discussed elsewhere.[6]

Tables II, III, and IV summarize the results of our lattice calculations. In Table I we give the values of the first 22 vertices V_n, correct to sixteen decimal places, for the $k=1$ theory (ϕ^6 supersymmetric theory). Table II gives the number of graphs that arise in order n. In Table III we list the numerical values of the coefficients A_n for $n=1,2,\ldots,11$.

TABLE II. Number of distinct, connected vacuum graphs in each of the first eleven orders of strong-coupling perturbation theory. In the ninth, tenth, and eleventh orders the graphs were enumerated with a computer. On a Ridge computer (whose speed is comparable with VAX 11/780) the calculations in orders eight, nine, ten, and eleven take 14 min, 1.3 h, 10.5 h, and 98.5 h of central processing unit time, respectively.

Order	Number of graphs
1	2
2	4
3	9
4	24
5	69
6	245
7	899
8	3792
9	17 257
10	85 990
11	461 357

In any calculation of this type involving thousands of graphs one must ensure that there are no calculational errors. There are several sources of calculational error: (i) omission of graphs, (ii) duplication of graphs, (iii) incorrectly computed symmetry numbers, (iv) error in computing the lattice sum for a graph, and (v) error in summing over graphs. We are absolutely positive that there are no errors in our calcula-

TABLE III. A_n, the negative of the sum of the values of the nth-order graphs for the $k=1$ theory. The value of each graph is computed by multiplying together the vertices, the symmetry number, and the lattice sum for that graph. Observe that at first A_n decays in magnitude with increasing n. However, after ninth order this pattern is reversed and $|A_n|$ increases with n.

n	A_n
1	0.056 025 349 067
2	−0.053 172 481 433
3	0.035 863 475 674
4	−0.021 275 014 204
5	0.011 775 536 702
6	−0.006 113 652 899
7	0.002 848 675 013
8	−0.001 017 881 459
9	0.000 058 214 376
10	0.000 366 925 659
11	−0.000 476 938 1
12	−0.000 423 (estimated)

tion because of the strategy we used to calculate A_n. We divided ourselves into four groups. Two groups working completely independently calculated $A_1, A_2, \ldots, A_8$ by hand. These two groups then tediously compared their graph-by-graph results for all 5044 graphs and these were then compared with the output of a simple computer program run on MACSYMA which generated the sum of the symmetry numbers in each order. Meanwhile, the third and fourth groups completely independently wrote and ran two computer programs which enumerated all graphs at each order, computed the symmetry numbers for each graph, computed the lattice sum for each graph, and then obtained A_n by summing over all graphs. Exact agreement between the two computer calculations and the two hand calculations was obtained for orders one through eight. On the basis of this agreement, we believe that the computer programs have no flaws. We then used both computer programs to generate A_9, A_{10}, and A_{11}, and obtained complete agreement.

The ninth-order computer calculation makes it quite clear that the strong-coupling graphs form a sharply peaked distribution centered at zero (see Fig. 1). Comparing the results with the distribution of seventh-order graphs plotted in Fig. 2 of Ref. 1 shows that the positive and negative tails of the distribution become smoother and more monotonic as the order of perturbation expansion increases. The ninth-order computer calculation also reveals a change in the nature of the strong-coupling lattice-expansion coefficients: Until eighth order, the magnitude of the coefficients falls slowly and regularly. However, in ninth order this pattern is abruptly changed. $|A_9|$ is 17.5 times smaller than $|A_8|$. Actually, the signal that the regular pattern is breaking first appears in eighth order. We have plotted the values of $(-1)^{n+1}A_n$ for the $k=1$ theory in Fig. 2 on a semilog graph. It is clear from this graph that the actual value of A_8 is lower by about 10% than the value one would have predicted by extrapolating from the first seven orders.

The values of $(-1)^{n+1}A_n$ are also given for the $k=2$ theory in the same plot. Now observe that the actual value of A_6 is lower than the value one would have predicted by extrapolating from the first five orders also by about 10%. The 10% discrepancy in sixth order is merely the first indication of a major change in the behavior pattern of the coefficients A_n. For the first six orders A_n decreases in magnitude and alternates in sign. However, A_7, in the $k=2$ theory, has the same sign as A_6, and therefore is not shown in Fig. 2.

In general, for an arbitrary k, the coefficients A_n of the lattice strong-coupling expansion show the same kind of abrupt deviation from a regular pattern at a critical order of perturbation theory—the higher the value of k, the earlier the change occurs. Before the critical value n, the perturbation coefficients decrease

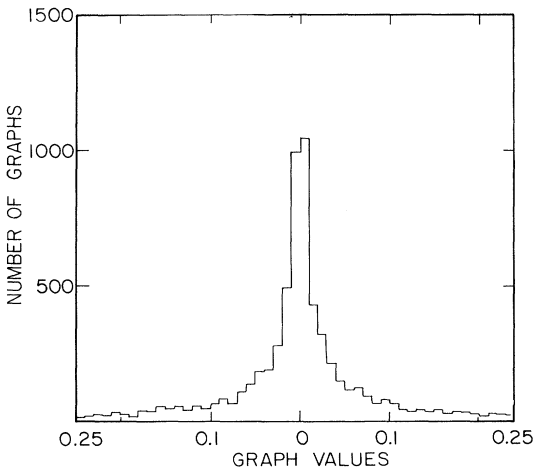

FIG. 1. The distribution of nonzero ninth-order graph values contributing to A_9 in the $k=1$ theory. There are a total of 17 257 graphs in ninth order of which 7570 are nonzero. All but 18% of these graphs lie in a range -0.25 to 0.25; 690 graphs evaluate to less than -0.25; 690 graphs evaluate to more than 0.25; and 6190 nonzero graphs lie in the range $[-0.25, 0.25]$. A comparison with Fig. 2 of Ref. 1 (order-seven graphs) shows that the distribution becomes smoother and more sharply peaked around the value zero as n increases.

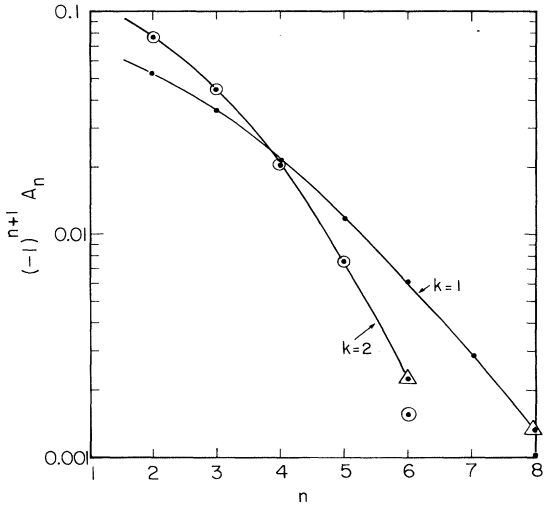

FIG. 2. The lattice strong-coupling series coefficients as a function of the order of perturbation theory n for both $k=1$ and $k=2$ theories. The dots represent the $k=1$ theory and the circled dots represent the $k=2$ theory. On this semilog plot it is clear that $(-1)^{n+1}A_n$ lie on smooth curves. The $k=1(k=2)$ theory first deviates from its projected value indicated by a triangle in order eight (order six). $A_9(k=1)=0.000\,06$ and $A_7(k=2)$ is negative, which shows that for sufficiently large n the behavior of the perturbation coefficients is undergoing a transition.

TABLE IV. The first ten continuum extrapolants $E_0^{(n)}$ for the ground-state energy density for the $k=1$ theory. These extrapolants were obtained from Eq. (4) by the extrapolation scheme described in Ref. 1. The observations that $E_0^{(7)}$ is larger than $E_0^{(6)}$ and that $E_0^{(8)}$ is still larger seem to indicate that the extrapolation scheme has failed. However, the superb results for $E_0^{(9)}$ and $E_0^{(10)}$ show that this is not the case. The temporary deviation from monotonic behavior corresponds to a transition in the large-order behavior of the strong-coupling lattice coefficients A_n.

n	$E_0^{(n)}/g^2$
1	17.677 266
2	178.488 60
3	279.919 01
4	28.748 400
5	7.996 587 7
6	3.731 282 1
7	3.991 788 3
8	4.243 49
9	2.167 24
10	1.946 49

in magnitude and alternate in sign. After the critical value the perturbation coefficients again alternate in sign, but are displaced by one order. Our best estimate for A_{12} is 0.000 423, which follows this pattern.

We use the extrapolation method described in Ref. 1 to calculate the continuum limit of the series in Eq. (4) representing E_0 for the $k=1$ theory. In Table IV we give the values of the first ten extrapolants. Observe that while the first six extrapolants are well behaved and appear to be approaching zero, the seventh extrapolant (which is computed from A_1 through A_8) is sensitive to the 10% deviation of A_8 from its extrapolated value and is larger than $E_0^{(6)}$. The eighth extrapolant to the ground-state energy density is still larger. However, once the sequence of perturbation coefficients A_n completes its transition and settles down to its new behavior, the extrapolants once again continue to decrease and approach zero. The ninth and tenth extrapolants clearly exhibit this behavior. It may be that the increasing size of $|A_n|$ yields a more slowly convergent sequence of extrapolants. This surmise must be checked by calculating A_{12}.

Finally, we wish to point out a startling regularity in the coefficients A_N first noticed in Ref. 1. From Table III it is clear that, having calculated the coefficients A_k for $k \leq N$, it is hard to imagine a predictive method of estimating A_{N+1} with high accuracy, especially in the region in which the sign oscillation misses an order. However, such a method does exist, accurate to within a few percent (except for the very small A_9, which is off by 9%). For example, the predicted value of A_{11} is $-0.000\,480$, while the actual value is $-0.000\,477$. The method is to expand $(\sum z^k A_{k+1})^{-4/N}$ to order Z^N, and adjust A_{N+1} so that the coefficient of Z^N vanishes. We do not understand why this method works, but we use it to give the estimated value for A_{12}.

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