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Defense of the Standard Quantum Limit for Free-Mass Position

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Measurements of the position x of a free mass m are thought to be governed by the standard quantum limit (SQL): In two successive measurements of x spaced a time τ apart, the result of the second measurement cannot be predicted with uncertainty smaller than $(\hbar\tau/m)^{1/2}$. Yuen has suggested that there might be ways to beat the SQL. Here I give an improved formulation of the SQL, and I argue for, but do not prove, its validity.

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Conventional wisdom[1,2] holds that in two successive measurements of the position x of a free mass m, the result of the second measurement cannot be predicted with uncertainty smaller than $(\hbar\tau/m)^{1/2}$, where τ is the time between measurements. This limit is called the *standard quantum limit* (SQL) *for monitoring the position of a free mass.*

The standard "textbook" argument for the SQL runs as follows. Suppose that the first measurement of x at $t=0$ leaves the free mass with position uncertainty $\Delta x(0)$. This first measurement disturbs the momentum p and leaves a momentum uncertainty $\Delta p(0) \geqslant \hbar/2\Delta x(0)$. By the time τ of the second measurement the variance of x (squared uncertainty) increases to

$$(\Delta x)^2(\tau) = (\Delta x)^2(0) + [(\Delta p)^2(0)/m^2]\tau^2 \geqslant 2\Delta x(0)\Delta p(0)\tau/m \geqslant \hbar\tau/m. \tag{1}$$

The standard argument views the SQL as a straightforward consequence of the position-momentum uncertainty principle $\Delta x(0)\Delta p(0) \geqslant \frac{1}{2}\hbar$.

Yuen[3] has pointed out a serious flaw in the standard argument. Between the two measurements the free mass undergoes unitary evolution. In the Heisenberg picture the position operator $\hat{x}$ evolves as

$$\hat{x}(t) = \hat{x}(0) + \hat{p}(0)t/m. \tag{2}$$

Thus the variance of x at time τ is given not by Eq. (1), but by

$$(\Delta x)^2(\tau) = (\Delta x)^2(0) + \frac{(\Delta p)^2(0)}{m^2}\tau^2 + \frac{\langle \hat{x}(0)\hat{p}(0) + \hat{p}(0)\hat{x}(0)\rangle - 2\langle \hat{x}(0)\rangle\langle \hat{p}(0)\rangle}{m}\tau. \tag{3}$$

The standard argument assumes implicitly that the last term in Eq. (3) is zero or positive. Yuen's point[3] is that some measurements of x leave the free mass in a state for which this term is negative. He calls such states *contractive states* because the variance of x decreases with time, at least for a while. As a result, the uncertainty $\Delta x(\tau)$ can be smaller than the SQL. Yuen[3,4] concludes that there are measurements of x that beat the SQL. My conclusion is different: The flaw lies in the standard argument, not in the SQL. In this Letter I give a new, heuristic argument for the SQL, formulate an improved statement of the SQL, and analyze a measurement model that supports the heuristic argument.

The heuristic argument is based on including the effect of the imperfect resolution σ of one's measuring apparatus. If the free mass is in a position eigenstate at the time of a measurement of x, then σ is defined to be the uncertainty in the result; thus, roughly speaking, the measuring apparatus can resolve positions that are more than σ apart. I assume that the measuring apparatus is coupled linearly to x, so that in general the variance of a measurement of x is the sum of σ^2 and the variance of x at the time of the measurement. Consider now two measurements of x at times $t=0$ and $t=\tau$, made with identical measuring apparatuses. In the absence of *a priori* knowledge, the result of the first measurement is completely unpredictable. Nonetheless, the first measurement does yield a value for x. Since this value does not tell one the position before the measurement, it is hard to see what one could mean by a "measurement of x with resolution σ" unless one means that the measurement determines the position immediately after the measurement to be within roughly a distance σ of the measured value.[5] Therefore, I assume that just after the first measurement, the free mass has position uncertainty $\Delta x(0) \leqslant \sigma$; this assumption implies that an *immediate* repetition of the same measurement would yield the same result within approximately the resolution σ. The variance of the second measurement ($t=\tau$) is given by

$$\Delta_2^2 = \sigma^2 + (\Delta x)^2(\tau) \geqslant (\Delta x)^2(0) + (\Delta x)^2(\tau) \geqslant 2\Delta x(0)\Delta x(\tau) \geqslant \hbar\tau/m. \tag{4}$$

According to this argument, *the SQL is a consequence of the uncertainty principle*

$$\Delta x(0)\Delta x(\tau) \geqslant \tfrac{1}{2}|\langle[\hat{x}(0),\hat{x}(\tau)]\rangle| = \hbar\tau/2m, \tag{5}$$

provided that $\sigma^2 \geqslant (\Delta x)^2(0)$. Contractive states do not vitiate this argument; even if the free-mass state after the first measurement is a contractive state such that $\Delta x(\tau) < (\hbar\tau/m)^{1/2}$, the SQL is valid.

Yuen uses a measurement model developed by Gordon and Louisell,[6] which includes the measuring-apparatus resolution. How then can he contend that it is possible to violate the SQL? The answer lies in the assumption $\sigma \geqslant \Delta x(0)$, which links the uncertainty Δ_2 in the second measurement to the position uncertainty $\Delta x(0)$ just after the first measurement. An easy way to circumvent this link, pointed out by Yuen,[4] is to use measuring apparatuses which are not identical. The first measurement, performed with an apparatus of poor resolution $\sigma_1 \geqslant \Delta x(0) >> (\hbar\tau/2m)^{1/2}$, is designed to leave the free mass in a contractive state such that $\Delta x(\tau) << (\hbar\tau/2m)^{1/2}$; the second measurement, performed with an apparatus of good resolution $\sigma_2 << (\hbar\tau/2m)^{1/2}$, has uncertainty $\Delta_2 = [\sigma_2^2 + (\Delta x)^2 \times(\tau)]^{1/2} << (\hbar\tau/m)^{1/2}$, which violates the SQL. Two such measurements should be regarded as a single measurement process, because in a sequence of measurements one would repeat the entire process, not the individual measurements separately. The first measurement is a preparation procedure for the second; it puts the free mass in a state that becomes a near eigenstate of position at the time of the second measurement. It is obvious that a measurement of x can have arbitrarily small uncertainty if one is allowed an arbitrary prior preparation procedure. Although the possibility of two such measurements may be important, more important is to sharpen the formulation of the SQL—to rule out this case to which the SQL clearly cannot apply.

With the preceding discussion in mind, I formulate the SQL as follows: *Let a free mass m undergo unitary evolution during the time τ between two measurements of its position x, made with identical measuring apparatuses; the result of the second measurement cannot be predicted with uncertainty smaller than $(\hbar\tau/m)^{1/2}$*. For this formulation to be true in general, the uncertainty in the second measurement must be understood to be an average uncertainty, averaged over the possible results of the first measurement; the averaging procedure is made explicit in the model considered below [see discussion preceding Eq. (14)]. This improved formulation of the SQL is still an important restriction, because it applies to a class of real experiments.[2] In these experiments one has available a particular technique for measuring x, which is used to make a sequence of measurements on a single free mass. The objective is to detect some external agent (e.g., a force) that disturbs x. The relevant question is how small a disturbance can be detected or, equivalently, how well one can predict the result of each measurement in the absence of the disturbance. The improved SQL addresses precisely this question. Notice that the improved SQL explicitly disallows any tinkering with the free mass during the interval between measurements; the free mass must evolve unitarily with no state preparation and no modification of its Hamiltonian.

Yuen would not agree even with the improved version of the SQL, because he believes that there are measurements that violate the assumption $\sigma \geqslant \Delta x(0)$.[3,4] Specifically, he suggests that there are ways to measure x which have good resolution $\sigma << (\hbar\tau/2m)^{1/2}$, but which leave the free mass in a contractive state with $\Delta x(0) >> (\hbar\tau/2m)^{1/2} >> \sigma$ such that $\Delta x(\tau) << (\hbar\tau/2m)^{1/2}$. Although contractive states are essential to this scheme, they are not enough to invalidate the SQL; also required are measurements of resolution σ that do not determine the position just after the measurement to be within σ of the measured value. Yuen states his suggestion in the notation of the Gordon-Louisell[6] formalism, which can describe formally measurements with $\Delta x(0) \geqslant \sigma$. The existence of this formal description, however, does not guarantee that such measurements can be realized. Gordon and Louisell simply assume the existence of certain measurements [Eqs. (22) and (23) of Ref. 6] without demonstrating that all such measurements can be realized. The measurements suggested by Yuen are among those for which no realization is known.[3,4]

I turn now to a simple model of measurements of x. Within the model one can demonstrate the validity of the SQL, but the model by no means provides a general proof. The model is, however, sufficiently general that it clarifies the meaning of the SQL and indicates where one might seek violations. I work in the Schrödinger picture; operators are denoted by caret.

The first task is to model the initial measurement of x; the model that I employ is like that of Arthurs and Kelly.[7] The free mass is coupled to a "meter," a one-dimensional system with coordinate Q and momentum P, which can be regarded as the first stage of a macroscopic measuring apparatus. The coupling is turned on from $t=-\tilde{\tau}$ to $t=0$ ($\tilde{\tau} \ll \tau$), it is described by an interaction Hamiltonian $K\hat{x}\hat{P}$ (K is a coupling constant), and it is treated in the impulse approximation (the coupling is so strong that the free Hamiltonians of the free mass and the meter can be neglected). The coupling correlates Q with x. When the interaction is turned off at $t=0$, one "reads out" a value for Q, from which one infers a value for x. The readout of Q can be viewed as an ideal measurement of Q made by the subsequent stages of the measuring apparatus.

At $t=-\tilde{\tau}$, just before the coupling is turned on, the free-mass wave function is $\psi(x)$, and the meter is prepared in a state with wave function $\Phi(Q)$. The total wave function is $\Psi_0(x,Q)=\psi(x)\Phi(Q)$; expectation values and variances with respect to $\Psi_0(x,Q)$ are distinguished by a subscript 0. For simplicity I assume that $\langle\hat{Q}\rangle_0=\langle\hat{P}\rangle_0=0$. At the end of the interaction time ($t=0$) the total wave function becomes

$$\Psi(x,Q)=\psi(x)\Phi(Q-x) \tag{6}$$

(units such that $K\tilde{\tau}=1$); expectation values and variances with respect to $\Psi(x,Q)$ are distinguished by having no subscript. The expectation value of Q at $t=0$ is $\langle\hat{Q}\rangle=\langle\hat{x}\rangle_0$. Thus the result of the first measurement—the inferred value of x—is the value $\bar{Q}$ obtained in the readout of Q. The expected result is $\langle\hat{Q}\rangle=\langle\hat{x}\rangle_0$, and the variance of the measurement is the variance of Q at $t=0$:

$$\Delta_1^2=(\Delta Q)^2=\sigma^2+(\Delta x)_0^2. \tag{7}$$

Here σ is the resolution of the meter, defined by $\sigma^2\equiv(\Delta Q)_0^2=\int dQ\,Q^2|\Phi(Q)|^2$. Notice that the variance (7) has the form assumed in Eq. (4)—a consequence of using an interaction Hamiltonian $K\hat{x}\hat{P}$ that is linear in $\hat{x}$.

The free-mass wave function $\psi(x|\bar{Q})$ just after the first measurement ($t=0$) is obtained (up to normalization) by evaluating $\Psi(x,Q)$ at $Q=\bar{Q}$:

$$\psi(x|\bar{Q})=\Psi(x,\bar{Q})/[P(\bar{Q})]^{1/2}=\psi(x)\Phi(\bar{Q}-x)/[P(\bar{Q})]^{1/2}, \tag{8}$$

$$P(Q)\equiv\int dx\,|\Psi(x,Q)|^2=\int dx\,|\psi(x)|^2|\Phi(Q-x)|^2. \tag{9}$$

Notice that $P(\bar{Q})$ is the probability distribution to obtain the value $\bar{Q}$ as the result of the first measurement. Expectation values and variances with respect to $\psi(x|\bar{Q})$ are distinguished by a subscript $\bar{Q}$.

During the time τ until the second measurement the free mass evolves unitarily. The second measurement is described and analyzed in exactly the same way as the first (assumption of identical measuring apparatuses). The expected result is the expectation value of x at time τ, which can be written as

$$\langle\hat{x}(\tau)\rangle_{\bar{Q}}=\int dx\,\psi^*(x|\bar{Q})[x+(\hbar\tau/im)(\partial/\partial x)]\psi(x|\bar{Q}), \tag{10}$$

$$\hat{x}(\tau)\equiv\hat{x}+\hat{p}\tau/m. \tag{11}$$

The result $\bar{Q}$ of the first measurement is known, and the meter wave function $\Phi(Q)$ is under one's control, but the free-mass wave function $\psi(x)$ before the first measurement is presumably not known. Nonetheless, I assume knowledge of $\psi(x)$ so that $\langle\hat{x}(\tau)\rangle_{\bar{Q}}$ can be calculated exactly. Then the unpredictability of the second measurement is characterized by its variance

$$\Delta_{2,\bar{Q}}^2=\sigma^2+[\Delta x(\tau)]_{\bar{Q}}^2, \tag{12}$$

$$[\Delta x(\tau)]_{\bar{Q}}^2=\int dx\,\psi^*(x|\bar{Q})[x+(\hbar\tau/im)(\partial/\partial x)-\langle\hat{x}(\tau)\rangle_{\bar{Q}}]^2\psi(x|\bar{Q}). \tag{13}$$

A simple case, corresponding to the argument leading to Eq. (4), occurs when one has almost no *a priori* knowledge about x before the first measurement—i.e., when $\psi(x)=|\psi(x)|e^{i\theta(x)}$ is such that $|\psi(x)|$ varies slowly on the scale set by σ. Then one finds that $\psi(x|\bar{Q})=\Phi(\bar{Q}-x)e^{i\theta(x)}$, which implies $\langle\hat{x}\rangle_{\bar{Q}}=\bar{Q}$ and $(\Delta x)_{\bar{Q}}^2=\sigma^2$;

the variance (12) of the second measurement becomes

$$\Delta^2_{2,\bar{Q}} = (\Delta x)^2_{\bar{Q}} + [\Delta x(\tau)]^2_{\bar{Q}} \geq |\langle [\hat{x},\hat{x}(\tau)]\rangle_{\bar{Q}}| = \hbar\tau/m.$$

An arbitrary $\psi(x)$ requires more care. It is possible to find $\psi(x)$ and $\Phi(Q)$ such that $\Delta^2_{2,\bar{Q}} < \hbar\tau/m$ for some values of $\bar{Q}$. Since one cannot control the outcome of the first measurement, a reasonable way to characterize the unpredictability of the second measurement is to average $\Delta^2_{2,\bar{Q}}$ over all values of $\bar{Q}$, weighting each value by its probability $P(\bar{Q})$[8]:

$$\Delta^2_2 = \int d\bar{Q}\, P(\bar{Q})\Delta^2_{2,\bar{Q}} = \sigma^2 + \langle [\hat{x}(\tau) - \langle\hat{x}(\tau)\rangle_{\hat{Q}}]^2\rangle, \tag{14}$$

$$\langle [\hat{x}(\tau) - \langle\hat{x}(\tau)\rangle_{\hat{Q}}]^2\rangle = \int dx\, d\bar{Q}\, \Psi^*(x,\bar{Q})[x + (\hbar\tau/im)(\partial/\partial x) - \langle\hat{x}(\tau)\rangle_{\bar{Q}}]^2\Psi(x,\bar{Q}). \tag{15}$$

The notation $\langle\hat{x}(\tau)\rangle_{\hat{Q}}$ emphasizes that $\langle\hat{x}(\tau)\rangle_{\hat{Q}}$ is a function of the operator $\hat{Q}$. The average variance of x just after the first measurement satisfies

$$\int d\bar{Q}\, P(\bar{Q})(\Delta x)^2_{\bar{Q}} = \langle(\hat{x} - \langle\hat{x}\rangle_{\hat{Q}})^2\rangle = \sigma^2 - \langle(\hat{Q} - \langle\hat{x}\rangle_{\hat{Q}})^2\rangle \leq \sigma^2, \tag{16}$$

$$\langle(\hat{x} - \langle\hat{x}\rangle_{\hat{Q}})^2\rangle = \int dx\, d\bar{Q}\, (x - \langle\hat{x}\rangle_{\bar{Q}})^2 |\Psi(x,\bar{Q})|^2. \tag{17}$$

Equation (16) is the analog of the assumption $\sigma \geq \Delta x(0)$ in the heuristic argument leading to Eq. (4). By noting that

$$\langle\langle\hat{x}\rangle_{\hat{Q}}\rangle = \int d\bar{Q}\, P(\bar{Q})\langle\hat{x}\rangle_{\bar{Q}} = \langle\hat{x}\rangle, \quad \langle\langle\hat{x}(\tau)\rangle_{\hat{Q}}\rangle = \int d\bar{Q}\, P(\bar{Q})\langle\hat{x}(\tau)\rangle_{\bar{Q}} = \langle\hat{x}(\tau)\rangle,$$

one can write the inequality

$$\begin{aligned}\Delta^2_2 &\geq [\Delta(x - \langle\hat{x}\rangle_Q)]^2 + [\Delta(x(\tau) - \langle\hat{x}(\tau)\rangle_Q)]^2 \\ &\geq |\langle[\hat{x} - \langle\hat{x}\rangle_{\hat{Q}}, \hat{x}(\tau) - \langle\hat{x}(\tau)\rangle_{\hat{Q}}]\rangle| = |\langle[\hat{x},\hat{x}(\tau)]\rangle| = \hbar\tau/m.\end{aligned} \tag{18}$$

Thus the average variance obeys the SQL.

Both the heuristic argument and the model just considered require an essential assumption—that the measuring apparatus is coupled linearly to x. Within the context of a linear coupling, the model is quite general, since it allows the meter to be prepared in any state. Linear coupling does apply to the specific case Yuen describes in Refs. 3 and 4, which involves Gaussian free-mass contractive states that he calls "twisted coherent states"; any nonlinear coupling to x would destroy the Gaussian character of these states. To seek violations of the SQL, one should consider nonlinear couplings—e.g., $Kf(\hat{x})\hat{P}$. A word of caution: The interaction $Kf(\hat{x})\hat{P}$ describes directly measurements of the quantity $y = f(x)$; interpreting and analyzing such measurements as measurements of x is difficult.

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[7]E. Arthurs and J. L. Kelly, Jr., Bell. Syst. Tech. J. **44**, 725 (1965).

[8]When both $|\psi(x)|^2$ and $|\Phi(Q)|^2$ are Gaussian functions, Eq. (8) shows that $(\Delta x)^2_{\bar{Q}} = [(\Delta x)_0^{-2} + \sigma^{-2}]^{-1} \leq \sigma^2$; thus, in this case, the SQL follows directly from the argument leading to Eq. (4), with no need for any averaging.