Vorton Method in Three-Dimensional Hydrodynamics

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The vorton method, developed earlier by one of us, is applied to the problem of the mutual penetration of two and four vorton rings. (A vorton is a vortex singularity.) Abrupt jumps of vorticity as well as major changes in the energy spectrum are observed in the process of destruction of vorton rings and in the instability of a system of vortons near collapse.

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In an earlier paper by one of us,[1] the generalized dynamics of three-dimensional vortex singularities (vortons) was introduced. The vorton method, simultaneously, gives a visual physical model and a method to calculate three-dimensional vortex dynamics. In Ref. 1 some analytical results were obtained. In this paper we investigate some more complicated problems numerically.

The velocity field induced by an isolated vorton in unbounded space has the form[1]

$$v_i^{(\alpha)}(t,\mathbf{x}) = -\frac{\epsilon_{ijk}(x_j - x_j^{(\alpha)})\gamma_k^{(\alpha)}}{4\pi|\mathbf{x}-\mathbf{x}^{(\alpha)}|^3}, \tag{1}$$

where ϵ_{ijk} is the unit antisymmetric tensor, and $\mathbf{x}^{(\alpha)}(t)$ and $\boldsymbol{\gamma}^{(\alpha)}(t)$ are the components of position and intensity, respectively, of the vorton labeled α.

When we use a special Lagrangian form of the equations of ideal incompressible flow, choose appropriate initial conditions, and apply a projection procedure (without any approximation), there results a closed system of ordinary differential equations for the coordinates and intensities of a system of N vortons[1]:

$$\begin{aligned} \dot{x}_i^{(\alpha)} &= -\frac{1}{4\pi}\epsilon_{ijk}\sum_{\beta=1}^{N}{}' r_{\alpha\beta}^{-1} n_j^{(\alpha,\beta)}\gamma_k^{(\beta)}, \\ r_i^{(\alpha,\beta)} &= x_i^{(\alpha)} - x_i^{(\beta)}, \\ r_{\alpha\beta} &= |\mathbf{r}^{(\alpha,\beta)}|, \\ n_i^{(\alpha,\beta)} &= r_i^{(\alpha,\beta)} r_{\alpha\beta}^{-1}, \end{aligned} \tag{2}$$

$$\dot{\gamma}_i^{(\alpha)} = -\frac{1}{4\pi}\epsilon_{ijk}\sum_{\beta=1}^{N}{}' r_{\alpha\beta}^{-3}(\gamma_j^{(\alpha)} - 3n_j^{(\alpha,\beta)} n_m^{(\alpha,\beta)}\gamma_m^{(\alpha)})\gamma_k^{(\beta)}, \tag{3}$$

where the dot indicates time derivative and primed sums omit the $\alpha=\beta$ term. Equation (3) reflects the phenomenon of vortex stretching in three-dimensional flow. These vorton equations differ from earlier studies of three-dimensional discrete vortex dynamics (see the review by Leonard[2]) in that there is no need for assumptions about a finite core or finite viscosity to establish a well-posed dynamical system. The vorton method is efficient computationally, especially when we consider that programs are easily vectorized and, as will be shown, four vortons are enough for a reasonable description of a vortex ring.

By analogy with the spectral description of linear vortices,[3] we can derive the energy spectrum of a vorton system. From (1),

$$\begin{aligned} \tilde{v}_i^{(\alpha)}(\mathbf{k}) &= \int v_i^{(\alpha)}(\mathbf{x}) e^{i\mathbf{k}\cdot\mathbf{x}} d^3x \\ &= -ik^{-2}\epsilon_{ijl}k_j\gamma_l^{(\alpha)} e^{i\mathbf{k}\cdot\mathbf{x}^{(\alpha)}}. \end{aligned} \tag{4}$$

The spectral density of the energy is determined by

$$E(k) = \sum_{\alpha,\beta} E_{\alpha,\beta}(\mathbf{k}), \tag{5}$$

$$E_{\alpha,\beta}(k) = \tfrac{1}{2}(2\pi)^{-3}\int ds_k\, \tilde{v}_i^{(\alpha)}(\mathbf{k})\tilde{v}_i^{(\beta)}(-\mathbf{k}). \tag{6}$$

The integral in (6) is taken over a sphere S_k of radius k in Fourier space. If the integral $\int_0^\infty E(k)\,dk$ converges, then

$$\mathscr{E} = \int_0^\infty E(k)\,dk = \tfrac{1}{2}\int_0^\infty v_i^2(\mathbf{x})\,d^3x, \qquad v_i(\mathbf{x}) = \textstyle\sum_\alpha v_i^{(\alpha)}(\mathbf{x})$$

would represent the energy of the system. However, the self-energy of a vorton is infinite. Thus it makes sense to define the interaction energy

$$\mathscr{E}_{\text{int}} = \int_0^\infty E_{\text{int}}(k)\,dk, \quad E_{\text{int}}(k) = \textstyle\sum'_{\alpha\beta} E_{\alpha\beta}(k). \tag{7}$$

Substituting (4) into (5)–(7), we obtain

$$E(k)=E_0+E_{\rm int}(k), \quad E_0=(1/6\pi^2)\sum_\alpha \gamma_i^{(\alpha)}\gamma_i^{(\alpha)}, \tag{8}$$

$$E_{\rm int}(k)=\frac{1}{2\pi^2}\sum_{\alpha,\beta}[\phi_1(kr_{\alpha\beta})\gamma_i^{(\alpha)}\gamma_i^{(\beta)}+\phi_2(kr_{\alpha\beta})\gamma_i^{(\alpha)}n_i^{(\alpha,\beta)}\gamma_j^{(\beta)}n_j^{(\alpha,\beta)}], \tag{9}$$

$$\phi_1(z)=z^{-3}[(z^2-1)\sin z+z\cos z], \quad \phi_2(z)=z^{-3}[(3-z^2)\sin z-3z\cos z], \tag{10}$$

$$\mathscr{E}_{\rm int}=\frac{1}{8\pi}\sum_{\alpha<\beta} r_{\alpha\beta}^{-1}(\gamma_i^{(\alpha)}\gamma_i^{(\beta)}+\gamma_i^{(\alpha)}n_i^{(\alpha,\beta)}\gamma_j^{(\beta)}n_j^{(\alpha,\beta)}). \tag{11}$$

Expressions (8)–(11) provide the basis for the numerical investigation of the evolution of the energy spectrum in time (Fig. 3). For the dynamical system (2),(3), $\mathscr{E}_{\rm int}$ is not an integral of motion (see also the results of our numerical experiments below). Because of three-dimensional stretching, the interaction energy can be transformed into self-energy of the vortex singularities.[1] This kind of inviscid dissipation of energy can support the realization of the inertial-range "$\frac{5}{3}$ law"[4] for the energy spectrum. This tendency is confirmed by our numerical experiments.

In Ref. 1, the vorton ring was introduced as the discrete analog of a continuous vortex ring. It is a system of identical vortons placed tangent to a circle at the vertices of a regular polygon. In such a configuration, the vorton intensities, according to (2) and (3), do not change, and the ring itself drifts perpendicular to its plane with a velocity that was calculated analytically in Ref. 1.

It is well known[5,6] that two identical vortex rings, placed one behind the other, will penetrate into one another. "The velocity field associated with the rear vortex ring has a radially outward component at the position of the front ring and so that radius of the front ring gradually increases. This leads to a decrease in its speed of travel, and there is a corresponding increase in the speed of the rear vortex, which ultimately passes through the larger vortex and in turn becomes the front vortex. The motion is then repeated."[6] Calculations of this effect are usually done under the assumption that the flow is axisymmetric, the rings are thin, and the distance between rings is small compared with their radius (see, for example, Klyatskin[7]). In this approximation the parameters of the rings are nearly invariant during their interaction.

The problem is much more complicated and interesting when the possible three-dimensional instability is taken into account and the distance between rings is comparable with their radius. This kind of configuration of rings develops approximately, for example, when an axisymmetrical body moves through a fluid. Here we apply the vorton method to this problem.

In Figs. 1–3 and in the discussion below, we present some results of three numerical experiments of interactions of two n-vorton rings with $n=4$, 12, and 24. The initial radius r_0, the distance h_0 between rings, and the specific intensity $\kappa_0=n\gamma_0/2\pi r_0$ is the same in all these experiments. The numerical results were obtained by the solution of (2) and (3), with a simple fourth-order Runge-Kutta scheme.

All three experiments give almost five full periods in which the rings mutually interpenetrate. After this,

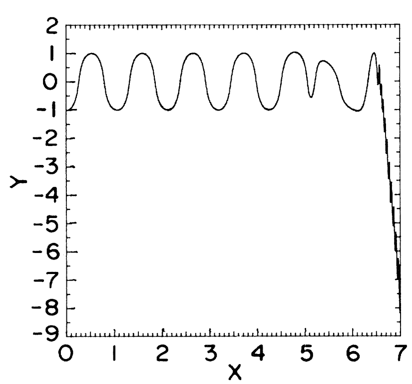

FIG. 1. The abscissa X is time; the ordinate Y is the distance Δh between two four-vorton rings in the direction of the axis of the rings (initially the first one is behind). $r_0=\frac{1}{2}\sqrt{2}$, $h_0=-1$, $n\gamma_0=40\sqrt{2}$.

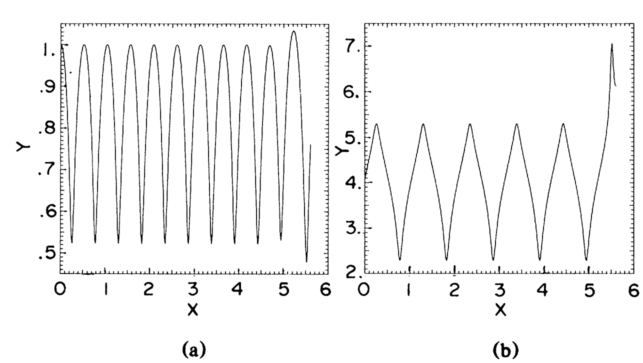

FIG. 2. The system of two twelve-vorton rings: (a) The abscissa X is time; the ordinate Y is the absolute distance between rings $[(\Delta h)^2+(\Delta r)^2]^{1/2}$ (Δr is the radial distance). (b) The abscissa X is time; the ordinate Y is the intensity of one of the vortons in the ring.

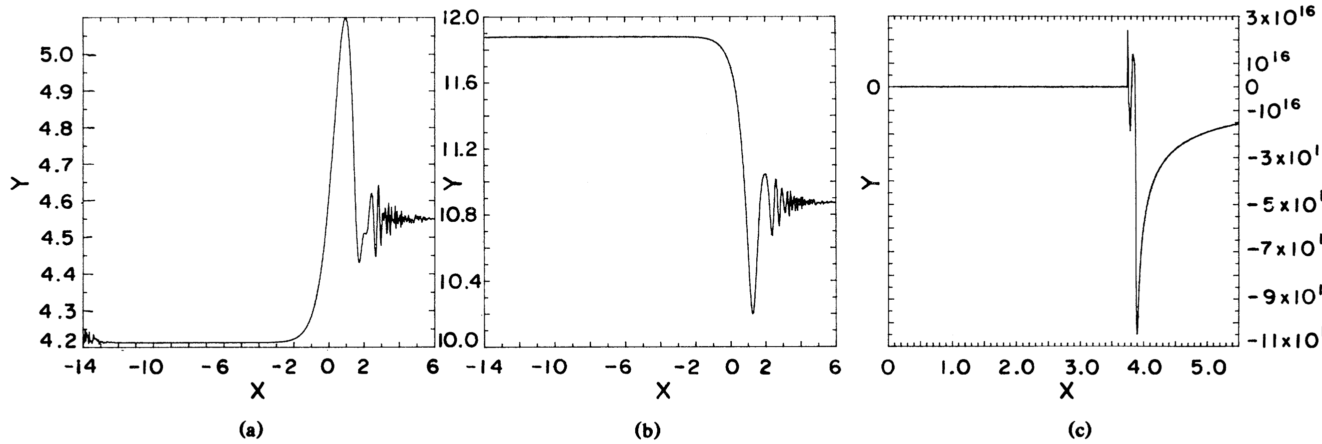

FIG. 3. The system of two 24-vorton rings: (a) The abscissa X is $\ln k$; the ordinate Y is the $E(k)$ at the initial moment. (b) The abscissa X is $\ln k$; the ordinate Y is $\ln E(k)$ at the moment of destruction of the rings. (c) The abscissa X is time; the ordinate Y is $d\mathscr{E}_{\rm int}/dt$, the inviscid dissipation of energy.

the system becomes unstable as three-dimensional components of vorton intensities appear with exponentially growing amplitudes and changing signs. The striking feature of these calculations is that the number of stable periods is insensitive to the number of vortons. Real vortex rings also become unstable after a few oscillations.

Let us consider some details of the numerical experiments. In the language of vortons, conservation of the specific intensity of a ring (by Kelvin's circulation theorem) means conservation of γ/r. This ratio changes by roughly 12%, 1.5%, and 0.7% for $n=4$, 12, and 24, respectively. The period of the interpenetration of rings decreases by 8.1% from $n=4$ to $n=24$. The "moment of inertia," which is proportional to $r_1^2+r_2^2$ (where r_1 and r_2 are the radii of the rings), is conserved with a precision of 0.1% for $n=24$. The absolute distance between the rings [Fig. 2(a)] is not constant (as is the case of $h_0 << r_0$) and has sharp minimums and smooth maximums that reflect the strong interaction of the rings when one is inside the other.

After the destruction of the rings, when vorton intensities abruptly increase, the energy spectrum [Fig. 3(b)] develops a power law k^{-n} with slope $n\sim 1.7$, which reflects an energy cascade toward small scales. The power-law spectrum can be seen for $0<\ln k<1$ (which is the natural interval for this system with a small number of vortons and a limited range of distances $r_{\alpha\beta}$) between two asymptotic regions of (8)–(10): A plateau exists when $k << 1$ and the oscillatory approach to another plateau when $k >> 1$. This agreement with the $\frac{5}{3}$ law may now be considered to be accidental because of the small number of vortons in this experiment. At the same time, we observed that the average cosine of the angle between vorton intensities jumps from -0.15 to $+0.5$. This indicates the existence of a coherent structure associated with the power spectrum.

The case of antiparallel rings moving toward each other is equivalent to the interaction of one ring with a rigid stress-free wall. The specific intensity is not conserved in this case, because asymptotically vortons of one ring cease to interact with each other, corresponding to the movement of vorton pairs[1] in infinite space.

We also carried out experiments with four twelve-vorton rings and initial distances

$$h_0^{(1,2)}=h_0^{(3,4)}=0.05h_0^{(2,3)}=-0.05r_0$$

(r_0 is the initial radius of rings). We can see (Fig. 4) the fast and stable mutual penetration of two neigh-

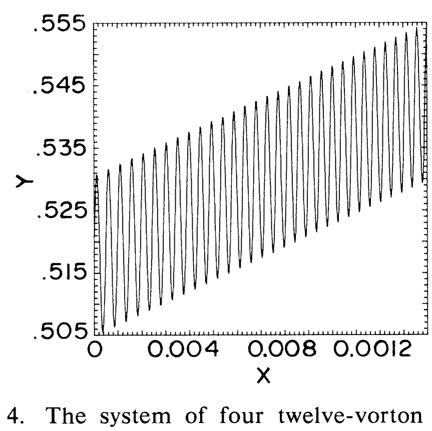

FIG. 4. The system of four twelve-vorton rings. The abscissa X is time; the ordinate Y is the distance between neighboring vortons in one ring, which is proportional to the radius of a ring (each maximum and minimum corresponds to a moment of interpenetration of the rings) ($n\gamma_0=120$, $h_0=-1$, $r_0=1$).

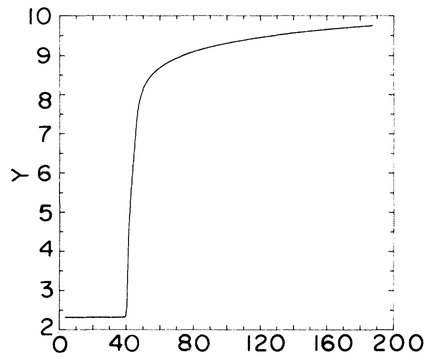

FIG. 5. Instability of the collapse of the three-vorton system. The abscissa X is time; the ordinate Y is $\ln|\gamma|$ (γ is the vorton intensity of one of the vortons).

boring rings superimposed on a slow interaction between the two pairs of rings. The specific intensity is conserved to within 0.4%. Further numerical experiments show that with appropriate initial conditions, systems of three and more vortex rings become chaotic.

Finally, we investigated different configurations of three vortons. It is interesting to study configurations that are initially close to collapsing,[1] because they involve strong interactions and can correspond to the physical mechanism of "vortex catastrophe" (sharp increase of integral vorticity $\int|\boldsymbol{\Omega}|d^3x$). We observed abrupt jumps of vorticity (Fig. 5) at the moment t_*, which correspond to the collapse of the unperturbed system. This circumstance is connected with the similarity of collapse.[1] When the intensity jumps, the spectrum develops a power law with slope $n \sim 1$. Shortly afterward, we observe the peak of inviscid dissipation of energy.

In later publications, the vorton method will be applied to study the stochastization of the vortex rings, the evolution of elliptic vortex rings, the expelling of vortex rings from horseshoe vortices in the boundary layer, and the detailed analysis of coherent structures connected with power-law energy spectra. From the derivation of Eqs. (2) and (3) given in Ref. 1, it follows that the vorton method can be used for a variety of physical systems including magnetohydrodynamics.

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