# Violation of Causality in Relativistic Quantum Theory?

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We show that states of systems which, in a very general sense, are approximately localized at time $t=0$ in a finite region, with exponentially bounded tails outside, violate Einstein causality at later times. Implications are discussed.

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Einstein causality means no propagation faster than the speed of light. Some time ago the present author[1] proved a theorem that a particle which at $t=0$ is localized with probability 1 in a finite volume of space immediately develops infinite "tails," irrespective of the particular notion of localization, be it in the sense of Newton-Wigner or others. For the Newton-Wigner case this phenomenon had already been observed previously,[2] and it also occurred in some models in which localization was expressed by means of a current density four-vector.[3] Later, an alternative proof of the theorem of Ref. 1 was given,[4] and it was extended to relativistic systems[5] and to quite general interactions.[6] The upshot of Ref. 6 was that this acausal behavior of strictly localized states is already entailed by very weak assumptions on the energy-momentum spectrum, by not much more than positivity of the energy.

At that time the obvious way of these difficulties with casuality was, of course, to assume that such a strict localization is not possible, that every particle already has tails, exponential say, to begin with. This would imply, for example, that no self-adjoint position operator exists, a consequence one could live with.[7]

In this paper I show that the situation for localization and causality is more complicated than previously thought. I will demonstrate that also relativistic particles or systems with exponentially bounded tails at $t=0$ violate Einstein causality at later times. Some of the questions this result raises will be discussed at the end of the paper.

*A consequence of finite propagation speed.*—Let $\psi_t = \exp\{-iHt\}\psi_0$ denote the state of a physical system at time $t$ ($\hbar=1$). Using one of the basic principles of quantum mechanics we assume that the probability of finding the system at time $t$ inside a region $V$ is given by the expectation value of an operator, say $N(V)$. Clearly, $0 \leqslant N(V) \leqslant 1$, but no further properties are required.[8] We say that a system is localized with exponentially bounded tails at $t=0$ if the probability of finding it outside the ball $B_r$ of radius $r$ around the origin decreases at least exponentially for large $r$,

$$\langle \psi_0, N(R^3 \backslash B_r)\psi_0\rangle \leqslant K_1 \exp\{-K_2 r^k\} \qquad (1)$$

where $K_{1,2}(\psi_0)$ are constants and where we will consider the exponent $k=1$ for a single particle and $k=2$ for a system. By adjustment of the constants, Eq. (1) may be taken to hold for all $r \geqslant 0$.

Let us now assume that the propagation speed is finite, bounded by $c'$, say, where $c' \geqslant c$, and $c=1$ is the speed of light. We consider a situation as depicted in Fig. 1. $V$ is a region of diameter $r_0$, a ball say, $V_{\mathbf{a}}$ its translate by $\mathbf{a}$. Let $\psi_0$ be a state with exponentially bounded tails at $t=0$. Now consider a later time $t>0$ and let $a \equiv |\mathbf{a}| > c't + r_0$. Then the probability of finding the system at time $t$ in $V_{\mathbf{a}}$ is less than or equal to that of finding it outside the ball $B_{a-c't-r_0}$ at time 0. Hence, by Eq. (1), one has for all $|\mathbf{a}| > c't + r_0$

$$\langle \psi_t, N(V_{\mathbf{a}})\psi_t\rangle \leqslant K_1 \exp\{-K_2(|\mathbf{a}| - c't - r_0)^k\}. \qquad (2)$$

Let $U(\mathbf{a})$ be the translation operator. Then[9]

$$N(V_{\mathbf{a}}) = U(\mathbf{a}) N(V) U(\mathbf{a})^*. \qquad (3)$$

One may, if one wishes, adjust the constants $K_1$ and $K_2$ such that Eq. (2) holds for all $|a| \geqslant c't$ and with $r_0$ omitted. Schwarz's inequality, applied to $\langle N(V)^{1/2}\psi_0, N(V)^{1/2} U(-\mathbf{a})\psi_t\rangle$, then gives

$$|\langle \psi_0, N(V) U(-\mathbf{a})\psi_t\rangle| \leqslant K' \exp\{-K(|\mathbf{a}| - c't)^k\} \qquad (4)$$

for all $|\mathbf{a}| \geqslant c't \geqslant 0$.

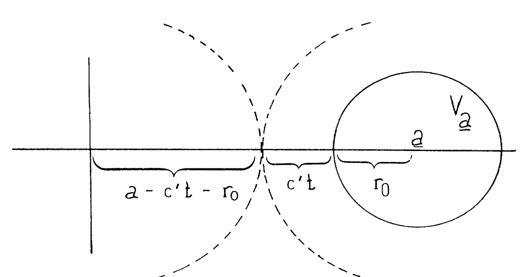


FIG. 1. Derivation of Eq. (2).

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This inequality is a consequence of the assumption of finite propagation speed $c'$ and of the assumption that $\psi_0$ has at most exponential tails. Equation (4) will be the starting point for the mathematical deductions below.

An alternative starting point leading to the same acausal behavior below is the following. *If* there is a $\phi_0$ such that $\psi_0 = N(V)^{1/2}\phi_0$ then, by Schwarz's inequality and Eq. (2), one has for $|\mathbf{a}| \geqslant c't \geqslant 0$

$$|\langle\psi_0, U(-\mathbf{a})\psi_t\rangle| \leqslant K'' \exp\{-K(|\mathbf{a}|-c't)^k\}. \quad (4')$$

This inequality means that the overlap between $\psi_t$ and the translated state $U(\mathbf{a})\psi_0$ at $t=0$ decreases at least exponentially in $|\mathbf{a}|-c't$. Equation (4′) can be obtained directly from Fig. 1 by an argument similar to the one leading to Eq. (2) if one believes—not quite reasonably—that widely different spatial localization of two states results in a small overlap. This avoids the introduction of the operator $N(V)$. But in my opinion the motivation of Eqs. (2) and (4) is physically more appealing and uses more basic principles of quantum mechanics.

*Violation of Causality.*—I will now prove the following theorem: A relativistic particle or system which at $t=0$ is localized with exponentially bounded tails ($k=1$ and $K_2 > 2m$ for a particle, $k=2$ for a system) violates causality at later times.

We first consider a free particle of mass $m \geqslant 0$ and arbitrary spin, which has exponentially bounded tails. We will show that the assumption of finite propagation speed $c'$ will lead, through Eq. (4) or Eq. (4′) with $k=1$, $K > m$, to a contradiction. We choose $V$ large enough so that $\langle\psi_0, N(V)\psi_0\rangle \neq 0$ and put $\phi \equiv N(V)\psi_0$ [in case of Eq. (4′) no $N(V)$ is needed and we put $\phi \equiv \phi_0$]. Then, for any fixed $t$, the function

$$f_t(\mathbf{x}) \equiv \langle\phi, U(-\mathbf{x})\psi_t\rangle \quad (5)$$

decreases as $\exp(-K|\mathbf{x}|)$ for large $|\mathbf{x}|$ and thus has a Fourier transform $\tilde{f}_t(\mathbf{p})$ which is analytic in the strip $|\mathrm{Im}\,\mathbf{p}| < K$. For a free particle one has $H \equiv P^0 = (\mathbf{p}^2+m^2)^{1/2}$, and thus in the momentum-space representation

$$f_t(\mathbf{x}) = (2\pi)^{-3/2}\int d^3p\, \exp(i\mathbf{p}\cdot\mathbf{x})\exp[-i(p^2+m^2)^{1/2}t]\bar{\phi}(\mathbf{p})\psi(\mathbf{p}),$$

where summation over spin indices is understood. Hence the integrand has to be analytic in $\mathbf{p}$ for $\mathrm{Im}|p| < K$. Because of the presence of the square root, this can happen for two distinct times only if the integrand vanishes identically. But then $f_t(\mathbf{x}) \equiv 0$ for all $\mathbf{x}$ and $t$, which implies for $\mathbf{x}=0$, $t=0$, that $\langle\psi_0, N(V)\psi_0\rangle = 0$ [or $\langle\psi_0,\psi_0\rangle = 0$ in the case of Eq. (4′)], a contradiction.

Now let us consider a general relativistic system with possible internal interaction. This case is more subtle since now the energy-momentum operator need only satisfy $P^\mu P_\mu \geqslant 0$. We will assume that the system has exponentially bounded tails, with exponent $k=2$ in Eqs. (4) and (4′). We need only consider translations in the direction of the $x^1$ axis. We set

$$\xi_{1,2} \equiv \tfrac{1}{2}\sqrt{2}(t \pm x^1),$$

$$P_{\mp} \equiv \tfrac{1}{2}\sqrt{2}(P^0 \mp P^1).$$

With $\phi$ as in Eq. (5), we define the function

$$f(\xi_1,\xi_2) \equiv \langle U(-x^1\mathbf{e}_1)\psi_t, \phi\rangle = \langle\psi_0, \exp\{i(P_-\xi_1 + P_+\xi_2)\}\phi\rangle. \quad (6)$$

Since $P_{\pm} \geqslant 0$, $f$ extends to an analytic function $f(z_1,z_2)$ for $\mathrm{Im} z_{1,2} > 0$ which is continuous and bounded by 1 for $\mathrm{Im} z_{1,2} \geqslant 0$.

I will show that Eq. (4) [or Eq. (4′)] with $k=2$ implies $f\equiv 0$, in particular $f(0,0)=0$, which means that $\langle\psi_0, N(V)\psi_0\rangle = 0$ [or $\langle\psi_0,\psi_0\rangle = 0$ in the case of Eq. (4′)], the same contradiction as before.

I employ Jensen's inequality,[10]

$$\log|g(0)| \leqslant (2\pi)^{-1}\int_0^{2\pi} d\theta \log|g(re^{i\theta})| \quad (7)$$

for a function $g(z)$ analytic in $|z| < R$, $R > r$. I define, for $|w_K| \leqslant 1$ and fixed arbitrary $z_\kappa^0 = \xi_\kappa^0 + iy_\kappa^0$, $y_\kappa^0 \geqslant 0$,[11]

$$g(w_1,w_2) \equiv f\left(iy_i^0\frac{1-w_1}{1+w_1} + \xi_1^0,\ iy_2^0\frac{1-w_2}{1+w_2} + \xi_2^0\right). \quad (8)$$

Then $g(0,0) = f(z_1^0,z_2^0)$, and $g$ is in each variable analytic on the unit disk and continuous including the boundary, except possibly at $w_K = -1$. I now apply Eq. (7) twice to $g(w_1,w_2)$, first for $w_1$ with $w_2$ fixed and then for $w_2$. In the limit $r \to 1$ we obtain (by Fatou's lemma)

$$\log|g(0,0)| \leqslant (2\pi)^{-2}\int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 \log|g(e^{i\theta_1},e^{i\theta_2})|.$$

A change of variable,

$$\xi_\kappa = \xi_\kappa^0 + iy_\kappa^0(1-\exp i\theta_\kappa)(1+\exp i\theta_\kappa)^{-1},$$

and insertion for $g$ from Eq. (8) then gives[12]

$$\log|f(z_1^0,z_2^0)| \leqslant \frac{y_1^0 y_2^0}{\pi^2}\int_{-\infty}^{\infty} d\xi_1 \int_{-\infty}^{\infty} d\xi_2 \frac{\log|f(\xi_1,\xi_2)|}{\left\{\prod_{\kappa=1}^{2}[(y_\kappa^0)^2 + (\xi_\kappa - \xi_\kappa^0)^2]\right\}^{-1}}. \quad (9)$$

Now with use of Eqs. (4) and (4′), a trivial computation shows that $\log|f|$ is bounded in region I of Fig. 2, up to a constant, by

$$-K(x^1-c't)^k \leqslant -K(-\xi_2)^k/\sqrt{2}. \qquad (10)$$

Therefore the integral over region I equals $-\infty$ for $k=2$ and then so does the whole integral because elsewhere $\log|f|$ may be estimated by 0. Thus $f(z_1^0,z_2^0)=0$ whenever, $\mathrm{Im}\,z_K^0>0$. Continuity then implies $f\equiv 0$. This completes the proof.[13]

*Discussion.*—The violation of casuality exhibited above expresses itself in essentially the following way. If a system has exponentially bounded tails at $t=0$ then, with finite propagation speed, it should still have such tails at later times, only shifted further out to infinity. As I have shown above this cannot be the case. Hence the state ("wave packet") spreads out to infinity faster than allowed by finite propagation speed. Conceivably, such a behavior might then also occur for systems which have only powerlike tails in their localization.

A possible way out of these difficulties may be to assume that states of particles or systems with exponentially bounded tails do not exist in the theory. But what if one had a similar behavior for powerlike tails? Another way out might be to assume that such well-localized states require infinite energy to prepare, i.e., that the expectation value of $H$ is infinite. This is a question that can be answered only in definite models, but to judge from the fact that already the Newton-Wigner position operator has states of finite energy which are strictly localized in bounded regions this seems not to be a very likely general possibility.

The exponential bound in Eq. (1) for the tails is not needed in this form for every $r \geqslant R_0$. It suffices to have it only for a sequence $r_n, r_n = nR_0$, say, since the probability must be monotone in $r$. Not needed and not assumed is commutativity of the $N(V)$'s—Eq. (4′) does not even contain them and Eq. (1) is a property of individual matrix elements. However, if one had finite propagation speed one would expect commutativity for disjoint regions. This in turn would facilitate the preparation of states with exponentially bounded tails. But then one gets a contradiction to finite propagation speed, a vicious circle.

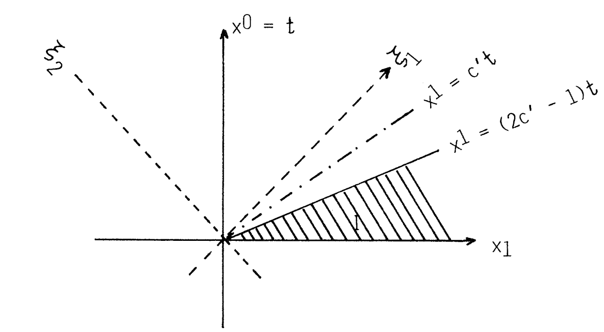


FIG. 2. Integration region for Eq. (9).

Full relativistic invariance did not enter in our derivation; only the energy-momentum (spectral) condition $P^\mu P_\mu \geqslant 0$ was used.[14] We therefore expect the results to carry over to more general circumstances, similarly as in Ref. 6.

This possible acausality is seen more as a problem of the underlying theory than as an experimentally verifiable prediction. For the latter, one would have to prepare, at $t=0$, a sufficiently large number of well-localized particles which do not interact, e.g., are sufficiently far apart, and then measure their arrival time at another location. If more arrive than allowed by the tails in the localization, one would have violation of casuality.

In conclusion I would like to point out that there may well be a connection between the above result and the Einstein-Podolsky-Rosen paradox and Bell's inequality.[15] For the latter the locality (causality) assumption plays a crucial role, and intuitively this leads to interpretational difficulties.

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[7]In this context, positive-operator–valued measures have been studied; cf., e.g., D. P. L. Castrigiano and R. W. Henrichs, Lett. Math. Phys. **4**, 169 (1980); for photons see K. Kraus, in *The Uncertainty Principle and Foundations of Quantum Mechanics,* edited by W. C. Price and S. S. Chissick (Wiley, London, 1977), p. 293.

[8]This operator is not needed if one starts directly from Eq. (4′) below. With a self-adjoint position operator, $N(V)$ would just be the projector in its spectral decomposition belonging to $V$.

[9]By translation invariance, the probability of finding any system $\phi$ in $V$ is the same as that of finding the translated system, $U(\mathbf{a})\phi$, in $V_{\mathbf{a}}$. Thus

$$\langle \phi, N(V)\phi\rangle = \langle U(\mathbf{a})\phi, N(V_{\mathbf{a}})U(\mathbf{a})\phi\rangle$$

which implies Eq. (3).

[10]Cf., e.g., H. Kneser, *Funktionentheorie* (Vadenhoeck and Ruprecht, Göttingen, 1958), p. 180; L. V. Ahlfors, *Complex Analysis* (McGraw-Hill, New York, 1953). To see this directly, let $G$ be a disk of radius $r$ around 0 with small disks around the zeros of $g$ taken out (semidisks if on $|z|=r$). Then in $G$, $\ln|g(z)| = \mathrm{Re}\ln g(z) \equiv u$ and $\Delta u = 0$. Thus, by Gauss's theorem,

$$\begin{aligned} 0 &= \int_G \Delta u \, d^2x \\ &= \int_{\partial G} (\partial u/\partial n)\, ds \\ &= r(d/dr)\int_0^{2\pi} u(r e^{i\theta})\, d\theta - 2\pi n(r) \end{aligned}$$

where $n(r)$ is the number of zeros of $g$ in $|z| \leq r$ (counted half if on $|z| = r$). Thus $(d/dr)\int_0^{2\pi} d\theta\, u(re^{i\theta}) \geq 0$, and integrating this over $r$ from 0 to $r$ gives Eq. (7). One may also use subharmonicity of $\ln|g|$.

[11]In the arguments of $f$ one has maps from the unit disk onto the upper half plane.

[12]For the one-variable case see, e.g., J. B. Garnett, *Bounded Analytic Functions* (Academic, New York, 1981), p. 64.

[13]A corollary is a sort of lower bound for the cluster decay, i.e., a decay as fast as in Eqs. (4) and (4′) is not possible for the above constants.

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