Differences between Lattice and Continuum Percolation Transport Exponents

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We use a scaling analysis to estimate critical exponents for the electrical conductivity, elastic constants, and fluid permeability near the percolation threshold of a class of disordered continuum systems (Swiss-cheese models), where the transport medium is the space between randomly placed spherical holes. We find that the exponents are significantly *larger* than their counterparts in the standard discrete-lattice percolation networks, except for the case of electrical conductivity in two dimensions, where they are equal.

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In this note, we consider transport properties near the percolation threshold of a class of "Swiss-cheese" continuum models, where spherical holes are randomly placed in a uniform transport medium. We find that the exponents governing the behavior of electrical conductivity and elastic constants in such media can be quite different from the corresponding ones in the conventional discrete-lattice percolation models.[1-4] We obtain a new exponent also for fluid permeability in the pore space of a system of random, overlapping spherical grains. These results may be contrasted with the exponents for *geometrical* percolation properties, such as the correlation length exponent ν, which have been previously confirmed by simulations to be the same for these models as for ordinary lattice percolation.[5,6]

The results of our analysis are summarized in Table I for the Swiss-cheese models in two and three dimensions. The exponents $\bar{t}$ and $\bar{f}$ are defined by the assumption that the macrsocopic electrical conductivity Σ and the shear modulus N vanish as the volume fraction q of holes approaches a critical value q_c, according to the power laws $\Sigma \sim (q_c - q)^{\bar{t}}$, and $N \sim (q_c - q)^{\bar{f}}$. The fluid permeability k, defined as the volume-rate of fluid flow through the space between the random spheres, under a unit macroscopic pressure gradient, will clearly vanish at the same value q_c of the sphere-volume fraction, and is assumed also to follow a power law, $k \sim (q_c - q)^{\bar{e}}$. We find, in two dimensions, that the conductivity exponent $\bar{t}$ is the same as the exponent t for a standard lattice resistor network at percolation, but that the exponent $\bar{f}$, for the present model is significantly larger than the exponent f of a lattice model with both bond-stretching and bond-bending elastic forces, which was recently studied.[1-4] For the *three*-dimensional Swiss-cheese model, we find that *both* $\bar{t}$ and $\bar{f}$ are larger than the corresponding discrete percolation exponents. Moreover, while the permeability and conductivity exponents are identical to each other in the standard lattice percolation model, we find that $\bar{e}$ is dramatically larger than $\bar{t}$ in our continuum model, in both two and three dimensions.

In our analysis, we follow previous authors[5,7] in mapping the continuum model onto a type of discrete random network. Unlike the standard discrete percolation problem, however, we must employ a *continuous* distribution of bond strengths, and our analysis shows that in many cases there is a large probability density for finding a small bond strength. It has been known for some time that such a distribution can lead to an increase of the conductivity exponent.[8,9] Here we examine the elasticity, as well as the conductivity and permeability exponents, by considering the contribution of the "singly connected bonds" in the "nodes-links-blobs" picture of the percolation backbone,[10] similar to the analysis of Kantor and Webman for the lattice elasticity problem.[2] Although our method is different from those in Refs. 8 and 9, the various methods lead to rather similar results, at least in the conductivity case.

The mapping of the Swiss-cheese model onto a discrete random network was described by Elam, Kerstein, and Rehr,[5] and is illustrated in Fig. 1 for the two-dimensional case of random circular holes punched in a conducting, elastic sheet. In higher dimensions, the construction corresponds to a Voronoi tesselation, and the bonds are the edges of the Voronoi polyhedra. In two dimensions, a bond is present if the two neighboring holes do not overlap, but the "strength" of the bond i depends crucially on the channel width δ_i [Fig. 1(b)]. It is important to note that δ_i has a continuous probability distribution $p(\delta)$ which approaches a *finite* limit $p(0)$, for $\delta \rightarrow 0^+$. Although there will clearly be some short-range correlations between the values of δ_i on nearby bonds, these correlations should not affect the finiteness of the distribution for $\delta \rightarrow 0$, and we shall ignore such correlations entirely.

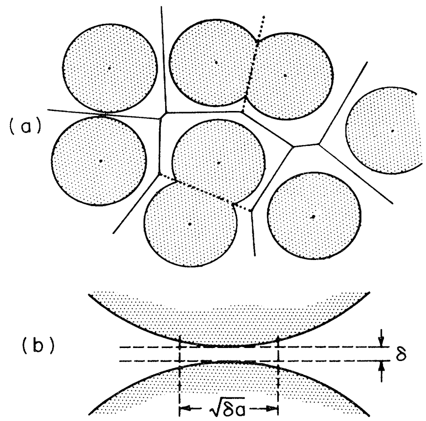

FIG. 1. Swiss-cheese model in two dimensions. Straight lines in part (a) show the bonds of the superimposed discrete network; dotted lines are the missing bonds. Dashed lines in part (b) outline a rectangular approximation to the illustrated narrow neck.

Next, we shall analyze the strength of a bond in the two-dimensional example. For the elasticity problem, the bond-bending force constant γ_i, associated with a narrow neck of width δ_i, is defined such that $\frac{1}{2}\gamma_i\theta^2$ is the energy necessary to bend the neck by a small angle θ. For small δ_i it is given, up to a constant of order unity, which we ignore, by

$$\gamma_i \sim Y_0\delta_i^{5/2}/a^{1/2}, \tag{1}$$

where Y_0 is the two-dimensional Young's modulus of the constituent material and a is the hole radius. This result may be understood if we approximate the neck by a thin rectangle of width δ_i and length $l_i \approx (\delta_i a)^{1/2}$, and use the classical result $\gamma_i \sim Y_0\delta_i^3/12l_i$ for a two-dimensional (2D) bent-beam problem[11] [see Fig. 1(b).] We note that the corresponding result for the 2D electrical conductivity is given by $g_i = \sigma_0\delta_i^{1/2}/a^{1/2}$, which has a much weaker dependence on δ_i.

In the three-dimensional Swiss-cheese model, the smallest cross section of a bond has roughly the shape of a triangle (see Fig. 2). By virtue of the Voronoi construction,[5,7] the centers of the three holes nearest a bond must be each at the same distance s_i from the bond, and the three sides of the triangular cross section are each proportional to the difference δ_i between s_i and the hole radius a, for small δ_i. Since this smallest cross section persists over a distance which is roughly $(\delta_i a)^{1/2}$, we find that the bond has an electrical conductance $g_i \sim \delta_i^{3/2}$, while the force constant γ_i for bond bending or for torsion is proportional to $\delta_i^{7/2}$. We expect that the distribution of s_i will be smooth in the vicinity of $s_i = a$, so that δ_i again has a finite probability density $p(0)$, in the limit $\delta_i \to 0$.[12]

Next we consider how these bonds are connected in the macroscopic system. In the nodes-links-blobs picture of percolation backbones, the conducting backbone of the infinite cluster is imagined to consist of a network of quasi-one-dimensional string segments (links), tying together a set of nodes whose typical separation is the percolation correlation length $\xi \sim (q_c - q)^{-\nu}$. Each string is supposed to consist of several sequences of singly connected bonds, in series with thicker regions, or blobs, where there are two or more conducting bonds in parallel.[10]

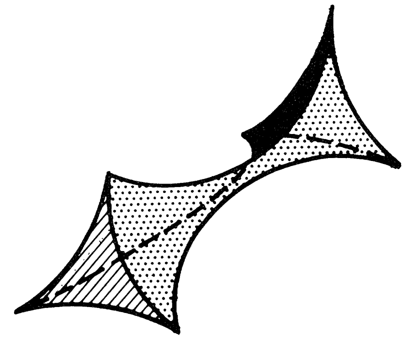

FIG. 2. Narrow portion of a bond, passing between three overlapping spherical holes, in the three-dimensional model.

To estimate the macroscopic conductivity, we ignore the resistance of the blobs, and approximate the conductance G of a string by

$$G^{-1} = \sum_{i=1}^{L_l} g_i^{-1}, \tag{2}$$

where the sum is restricted to the L_1 singly connected bonds on the string. It has been shown[10] that the typical value of L_1 is proportional to $(q_c - q)^{-1}$. In the conventional case, where each bond has *unit conductance,* the conductance of a string will be L_1^{-1}, and the conductivity of the network will be $\Sigma \sim \xi^{2-d}L_1^{-1}$, where d is the spatial dimension. Thus, this analysis predicts $t \approx 1 + (d-2)\nu \equiv t_1$, a result which slightly underestimates the true value of t in two and three dimensions.[1,10]

For the elastic problem, we define a force constant k for a string such that $\frac{1}{2}Ku^2$ is the energy cost to displace one end of the string by a small distance u, when the other end of the string is clamped in position and orientation. If one only considers the compliance of the singly connected bonds in the string, one finds

$$\frac{1}{K} = \sum_{i=1}^{L_1} \zeta_i^2/\gamma_i, \tag{3}$$

where γ_i is the bending force constant of bond i, and ζ_i, the moment arm of the ith bond, is a length of order ξ. If all bonds have the same bending constant, as is the case in the conventional lattice percolation model, we find $K \sim \gamma/L_1\xi^2$. Since the macroscopic elastic constants are proportional to $\xi^{2-d}K$, this implies the relation $f \approx 1 + d\nu \equiv f_1$, a result first obtained by Kantor and Webman.[2]

Now we must estimate a *typical*[13] value of the string conductance G or the force constant K for our case, where there is a distribution of bond strengths. If a string contains many singly connected bonds, we should be able to replace the sum in Eq. (2) or (3) by an integral over the probability distribution $p(\delta)$, provided that we properly control the contribution of the weakest bonds. In particular, using Eq. (1) we replace Eq. (3) by

$$\frac{1}{K} \approx \frac{a^{1/2}}{Y_0} \xi^2 L_1 \int_{\delta_{\min}}^{\infty} p(\delta)\, d\delta/\delta^x, \tag{4}$$

where $\delta_{\min}$ is the minimum value of δ for the singly connected bonds on this string, and $x = \frac{5}{2}$ for $d=2$ ($x=\frac{7}{2}$ for $d=3$, and Y_0 is replaced by the 3D Young's modulus). It may be seen that for large values of L_1, a typical value of $\delta_{\min}$ is $\approx \delta_0/L_1$, where $\delta_0 \equiv 1/p(0)$. {More generally,

$$\left[1 - \int_0^{\epsilon} p(\delta)\, d\delta\right]^{L_1} \approx e^{-\epsilon L_1/\delta_0}$$

is the probability that $\delta_{\min} > \epsilon$.} We see than that most of the time K is determined by the weakest singly connected bond on the string; i.e., that $K \sim \xi^{-2} L_1^{-x}$. If we use this estimate[12,13] to determine the shear modulus, via $N \sim K\xi^{2-d}$, we find $\bar{f} \approx d\nu + \frac{5}{2} = f_1 + \frac{3}{2}$ for $d=2$, and $\bar{f} \approx d\nu + \frac{7}{2} = f_1 + \frac{5}{2}$ for $d=3$.

For the electrical conductivity, we may apply a similar analysis to estimate the sum in Eq. (2). In $d=3$, we again find that the resistance of a string is determined by the weakest singly connected bond and the exponent $\bar{t}$ is $\nu + \frac{3}{2} = t_1 + \frac{1}{2}$. In $d=2$, however, where the bond conductance varies less rapidly than linearly with δ, the sum is not dominated by the weakest link; rather we have $G^{-1} \sim \bar{\rho} L_1$, where $\bar{\rho}$ is the mean resistance of a bond. Thus, we find that $\bar{t} = t$ in this case.

The difference between the permeability and conductivity exponents arises from the different behaviors of the bond strengths, for small δ_i. In three dimensions, the flow of a viscous fluid through a narrow channel like that in Fig. 2 is proportional to $\delta_i^4/l_1 = \delta_i^{7/2}/a^{1/2}$, while in a two-dimensional version of the model, the flow varies as $\delta_i^{5/2}/a^{1/2}$. Thus, we estimate the permeability exponent as $\bar{e} \approx \nu + \frac{7}{2} = t_1 + \frac{5}{2}$ for $d=3$, and $\bar{e} \approx \frac{5}{2} = t_1 + \frac{3}{2}$ for $d=2$.

Roberts and Schwartz[7] have studied numerically the electrical conductivity (and recently the permeability) of a porous rock, using a model similar to ours, except that the centers of their interpenetrating insulating spheres were chosen originally from a Bernal distribution, rather than completely at random. We would not expect this additional short-range correlation to affect the critical exponents. As noted by Roberts and Schwartz, however, the volume fraction of conductor in these models is very small ($\approx 3\%$) at the percolation threshold, and we expect that the critical exponent may be observable only for q very close to q_c. Roberts and Schwartz do not investigate this, but study instead a wide range of conducting-volume fractions above percolation. We note that the analysis of Roberts and Schwartz involved mapping onto a discrete network, similar to ours, with bond strengths determined by the cross-sectional area of the necks.

Wong, Koplik, and Tomanic[14] have studied a network of conducting pipes, in which all bonds on a regular network are present, but there is a wide distribution of pipe radii. The exponents of their model depend on parameters in the distribution, but they find typically a permeability exponent about twice the conductivity exponent, which is not far from our finding.

It must be emphasized, however, that other types of continuum models can lead to results which are very different from the ones that we have found. For example, if the conducting elements in a Swiss-cheese model are taken to be the interpenetrating spheres, rather than the space between the spheres, the conductivity exponent $\bar{t}$ will be the same as the lattice exponent t in $d=3$ as well as $d=2$, and the permeability exponent $\bar{e}$ would be approximately $t + \frac{1}{2}$ in $d=3$. According to our theoretical analysis, the two-dimensional system constructed by Smith and Lobb,[15] using a laser speckle pattern, should also have $\bar{t} = t$, as would a hypothetical $d=3$ generalization. The elastic model studied by Benguigui,[4] with circular holes at randomly selected sites of a regular lattice, has no narrow necks, and is expected to have the same exponents as a lattice elastic model. The differences among these various continuum models arise from the differences in the probability distribution and geometry of the narrowest channels, and all can be analyzed by the methods of the present Letter.

We remark, in closing, that the exponents derived in the text, by considering only the singly connected bonds on the strings (links) of the percolating backbone are presumably lower bounds to the true ex-

TABLE I. Estimates of the differences between the transport percolation exponents in the Swiss-cheese continuum model and the corresponding exponents on a discrete lattice. (See text for definitions.)

d	Conductivity ($\bar{t}-t$)	Elasticity ($\bar{f}-f$)	Permeability ($\bar{e}-t$)
2	0	$\frac{3}{2}$	$\frac{3}{2}$
3	$\frac{1}{2}$	$\frac{5}{2}$	$\frac{5}{2}$

ponents of the various models. On the other hand, if the exponents t and f in Table I are defined as the exact lattice values,[1] then the exponent *differences* $(\bar{t}-t)$, $(\bar{f}-f)$, and $(\bar{e}-t)$ listed in the table turn out to be *upper bounds* for the Swiss-cheese continuum model. These bounds can be established by an extension of the "variational method" employed in Ref. 8.

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[1]See, for example, S. Feng, P. N. Sen, B. I. Halperin, and C. J. Lobb, Phys. Rev. B **30**, 5386 (1984), and references therein. Numerical values of discrete lattice exponents defined in the text are the following: for $d=2$, t_1-1, $t\approx 1.3$, and $f\approx f_1\approx 3.7$; for $d=3$, $t_1\approx t\approx 1.9$ and $f\approx f_1\approx 3.6$.

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[11]See, for example, R. V. Southwell, *An Introduction to the Theory of Elasticity* (Dover, New York, 1969), Chap. VI.

[12]More detailed arguments will be given elsewhere.

[13]More precisely, we should define the representative value of G as the value G_c such that a fraction p_c of the strings have $G<G_c$, where p_c in turn is the fraction of strings that can be randomly cut before the backbone ceases to conduct. Strings with $G \ll G_c$ or $G \gg G_c$ cannot affect the conductivity of the multiply connected backbone, following V. Ambegaokar, B. I. Halperin, and J. S. Langer, Phys. Rev. B **4**, 2612 (1971). If p_c were equal to $\frac{1}{2}$, then G_c would be the *median* of G; other estimates for p_c change G_c somewhat, but do not change the exponent $\bar{t}$. We choose K_c similarly.

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