Finite-Size Studies of the Incompressible State of the Fractionally Quantized Hall Effect and its Excitations

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The incompressible states of interacting two-dimensional electrons in a high magnetic field has been studied by finite-size calculations in the spherical geometry. The excitation spectrum at $\frac{1}{3}$ Landau-level filling is described, and the Laughlin-Jastrow character of the Coulomb-interaction ground state is unambiguously confirmed. As the interaction is varied, a transition to a gapless compressible state is observed.

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The observation[1] in two-dimensional electron systems at high magnetic fields of quantization of the Hall conductance $\sigma^{xy}=\nu e^2/h$ with *fractional* values of ν has emphasized the collective origin of the quantized Hall effect (QHE), which may generally be associated with *incompressibility* of the electronic state. While incompressibility is a simple consequence of the Pauli principle in the case of integral Landau-level fillings, its existence at various fractional fillings (implied by the observations of a QHE) must be a result of repulsive interactions between the electrons.

This Letter reports a study of the interacting two-dimensional electron fluid at Landau-level filling $\nu=\frac{1}{3}$ by numerical diagonalization of systems of up to eight particles in spherical geometry.[2] A detailed picture of the incompressible state responsible for the $\nu=\frac{1}{3}$ QHE, plus its collective excitation and fractionally charged defects, has been obtained. Our calculations directly test the Jastrow-function model of the ground state introduced by Laughlin.[3] We believe that we have definitively confirmed its validity for the Coulomb interaction at $\nu=\frac{1}{3}$, and established that the incompressibility giving rise to the fractional QHE is a consequence of a *strong short-range component* of the interaction potential.

The spherical geometry is particularly suited to the study of homogeneous fluid states in finite systems: It is the only geometry in which fully translationally and rotationally invariant states with finite particle number N occur. Periodic boundary conditions[4] are more appropriate for the study of lattice states.

Cyclotron motion of a charged particle moving on a sphere of radius R in a radial magnetic field B is formally equivalent to precession of a symmetric top with quantized internal angular momentum $\hbar S$, where the total flux $\Phi=4\pi R^2 B$ is an integral number $2S$ of flux quanta $\Phi_0=h/e$. Interacting particles in a partially filled Landau level with index n become formally equivalent to a partially filled nuclear shell with orbital angular momentum $l=S+n$. Within a level, the interaction is fully characterized[2] by a set of *pseudopotential* coefficients V_m, the energies of a pair of particles with relative angular momentum m, i.e., total angular momentum $2l-m$. (The set $\{V_m\}$—and the underlying physics—is very different from that of the nuclear problem.) We discuss only the lowest Landau level ($l=S$) with full spin polarization so that only V_m with odd m couple the particles.

On the sphere, the antisymmetric, rotationally invariant Laughlin-Jastrow (LJ) wave function for the incompressible state at $\nu=1/m$, m odd, is[2]

$$\Psi_m^{\rm LJ}=\prod_{i<j}(u_i v_j - v_i u_j)^m,$$

where

$$(u,v)=(\cos\tfrac{1}{2}\theta\, e^{i\phi/2}, \sin\tfrac{1}{2}\theta\, e^{-i\phi/2})$$

are spinor variables describing particle coordinates. As a function of (u_i,v_i), wave functions are polynomials[2] of degree $2S$: $\Psi_m^{\rm LJ}$ occurs at flux $2S=m(N-1)$. Its surface density is uniform, given by $(2\pi m l^2)^{-1}$ (i.e., $\nu=1/m$) in the limit $N\rightarrow\infty$, where from now on l is the "magnetic length" $(\hbar/eB)^{1/2}$. The radius R of the sphere is $l\sqrt{S}$.

We numerically diagonalized the N-particle Hamiltonian parametrized by $\{V_m\}$, deriving these from the pure Coulomb potential $e^2/4\pi\epsilon r$, taking the geometric (chord) distance as the interparticle separation. Energies are quoted in units $e^2/4\pi\epsilon l$.

The exact ground state at flux $2S=3(N-1)$ ($\nu=\frac{1}{3}$) was, as expected, an isotropic state with total angular momentum $L=0$, well separated from excited states at the same (N,S) by a gap. As an example, Fig. 1 shows the full spectrum with $N=7$, $S=9$, consisting of 1656 multiplets with L ranging from 0 to 42. Figure 2(a) shows the ground-state energy per particle for various N. The contribution from a neutralizing surface charge $-Ne$ has been included. Extrapolation to

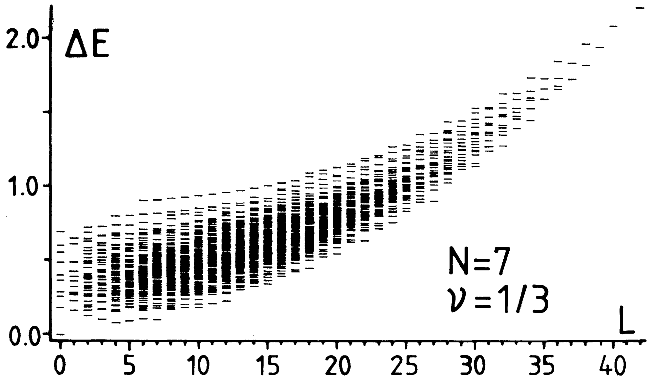

FIG. 1. The spectrum of 1656 multiplets (50 388 states) of the $N=7$ electron, $2S=18$ flux quanta system with Coulomb interactions, grouped by total angular momentum L. Energies (in units of $e^2/4\pi\epsilon l$) are shown relative to the incompressible ($\nu=\frac{1}{3}$) isotropic ($L=0$) ground state.

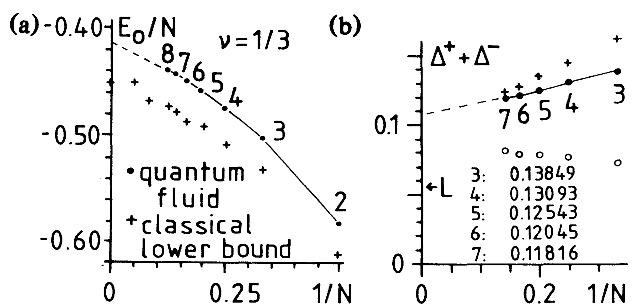

FIG. 2. (a) Ground-state energy per particle with a neutralizing background. The energy of classical point charges is a lower bound. (b) Creation energy $E_0^+ + E_0^- - 2E_0$ for a quasihole plus quasiparticle pair at infinite separation. Filled circles and legend: sequence with condensate neutralization only, energies compared at fixed field. Crosses: same, but energies compared at fixed surface area. Open circles: full neutralization, fixed field; this sequence will have a $1/\sqrt{N}$ dependence.

infinite N gives an energy -0.415 ± 0.005 per particle, in line with results from the plasma formulation[3] and the periodic geometry.[4] The difference between the exact value (e.g., $-0.450\,172$ at $N=6$) and the variational energy of the LJ state ($-0.449\,954$ at $N=6$) is not visible on the scale of Fig. 2(a).

The ground-state pair correlation for $N=6$ is compared with that of the LJ state in curve a of Fig. 3. They are extremely similar, the LJ state's projection on the ground state being over 99%. At short distances, the LJ state correlation vanishes as r^6; since on the sphere[2] this is the *only* state in the Hilbert space with no r^2 component, a weak r^2 component due to admixture of other states is found in the exact ground state (Fig. 3 inset). This has also been seen in periodic geometry.[4]

Dilation or contraction of the "incompressible" LJ state by variation of the magnetic field at fixed surface area produces fractionally charged defects.[3] Localization of this charge in the interior of the system far from edges requires an integral flux change, and the sphere has no edges. The elementary defects correspond to $S \to S \pm \frac{1}{2}$ at fixed N. The LJ defect wave functions[3] on the sphere are[2]

$$\Psi_m^+ = \prod_i (\partial/\partial v_i)\Psi_m^{\rm LJ}, \quad \Psi_m^- = \prod_i v_i \Psi_m^{\rm LJ}.$$

These have total angular momentum $L=\frac{1}{2}N$, and azimuthal angular momentum $M=\pm L$; both defects are localized at $\theta=0$. Their charge $q=\pm e/m$ is directly identified by the angular momentum quantum number L: A localized charge q in a radial magnetic field emitting flux Φ has intrinsic radial angular momentum $q\Phi/4\pi$. This gives $q/e = \lim_{N\to\infty}[\pm L/S] = \pm 1/m$, the limit being required as the defect is an extended object.

The actual ground state at $2S=3(N-1)\mp 1$ is indeed an isolated multiplet with $L=\frac{1}{2}N$ [Fig. 4(a)]. Charge-density profiles of the defects as a function of (chord) distance from their centers are shown in Fig. 3, curves b and c. The exact and the LJ-state results are again extremely similar. The model "quasihole" ($q=-\frac{1}{3}e$) defect has vanishing density at its center, and is the only state in the Hilbert space with this property. Weak admixture of other states leads to a small nonvanishing density at the center of the exact defect.

The fractional charge is visible in Fig. 3: Creation of a defect at fixed total charge means that the charge of the background LJ condensate must be depleted or augmented by $\frac{1}{3}e$. The asymptotic charge densities in

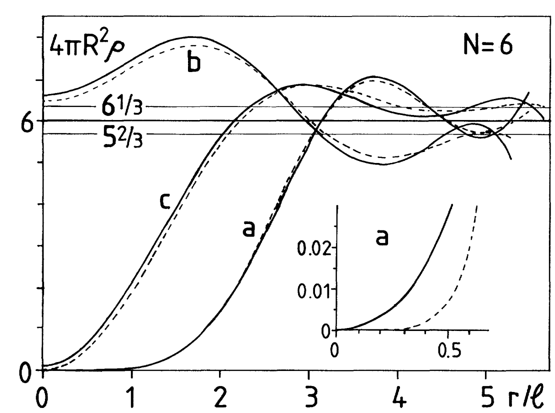

FIG. 3. (a) Ground-state pair correlation function for $\nu=\frac{1}{3}$, $N=6$. (b),(c) Density profiles of localized quasiparticle and quasihole defects. The condensate density ρ satisfies $4\pi R^2\rho=6$, $5\frac{2}{3}$, and $6\frac{1}{3}$, respectively. Filled curves, Coulomb interaction; broken curves, model Laughlin-Jastrow wave functions.

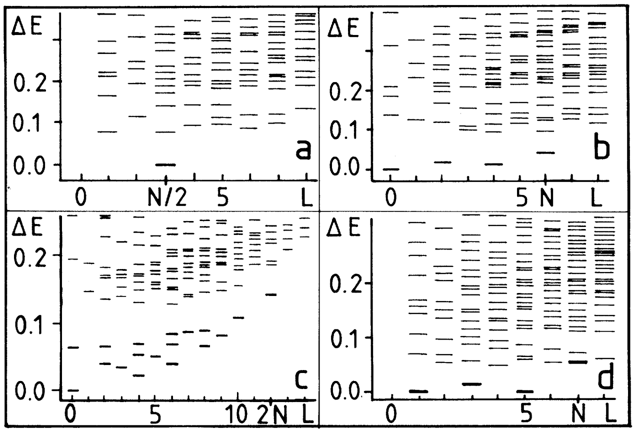

FIG. 4. Details from low-lying excitation spectra. (a) $N=6$, $2S=16$ ($\nu=\frac{1}{3}+$one quasihole). (b) $N=6$, $2S=17$ ($\nu=\frac{1}{3}+$two quasiholes). (c) $N=6$, $2S=19$ ($\nu=\frac{2}{7}$, or $\nu=\frac{1}{3}+$four quasiholes). (d) $N=7$, $2S=16$ ($\nu=\frac{1}{3}+$two quasiparticles).

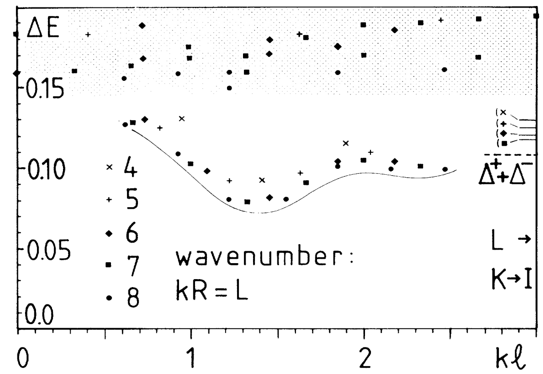

FIG. 5. Low-lying excitations at $\nu=\frac{1}{3}$, $N=4$, 5, 6, 7, and 8, showing the collective "quasiexciton" dispersion. The full line and shading indicating the continuum are a guide to the eye.

Fig. 3 do appear to satisfy $4\pi R^2\rho = \approx N \mp \frac{1}{3}$.

We estimate the gap $\Delta^+ + \Delta^-$ for the creation of a pair of defects at infinite separation as 0.105 ± 0.005 [Fig. 2(b)]—a factor of 2 larger than the plasma result[3] (shown as L). Various forms of the combination $E_0^+ + E_0^- - 2E_0$ of defect and pure state ground-state energies should converge to the gap as $N \to \infty$. To avoid $1/\sqrt{N}$ corrections, we include a neutralizing charge $-(N \mp \frac{1}{3})e$ *for the LJ condensate but not the defect* in calculating E_0^+ and E_0^-.

There are three proposed descriptions of the defect statistics: Bose,[2] Fermi,[5] and fractional.[6,7] These schemes are not in contradiction; each is internally consistent, and leads to the same hierarchy[2,5,6] of rational quantizations of the Hall effect. We view the Bose scheme[2] as the simplest description of defects produced at fixed fermion number N; in this picture, defects experience electron density as electrons experience magnetic-flux density.

The low-lying spectrum at $N=6$, $2S=17$ (two quasihole defects) is shown in Fig. 4(b). The two-defect states have $L=N$, $N-2$, $N-4,\ldots,$ consistent with defects being identical particles. In the Bose scheme,[2] these are the symmetric combination of two $L=\frac{1}{2}N$ defects.

The relative energies of the two-defect states with $L=N-m$, $m=0,2,4,\ldots,$ define defect pseudopotential coefficients $\tilde{V}_m$. From Fig. 4(b), it can be seen that the repulsive contact interaction $\tilde{V}_0$ is dominant. Figure 4(c) is in our view confirmation of the hierarchy picture[2,5,6]: It shows the low-lying states of a four-hole system, corresponding[2] to the $\nu=\frac{2}{7}$ state, an $m=2$ Bose LJ state of quasihole defects of the $\nu=\frac{1}{3}$ fundamental state. The content of the low-lying group of eighteen multiplets, well separated from higher levels, is as expected for four Bose $L=\frac{1}{2}N$ particles. This group, with a singlet ground state well below the others, resembles a miniature replica of the $\nu=\frac{1}{3}$ level pattern of Fig. 1. The low-lying states of Fig. 4(d) are two-quasiparticle states. Again, statistics and dominance of the contact repulsion are seen.

Figure 5 shows the low-lying excitation spectrum *at* $\nu=\frac{1}{3}$. Neutral-particle wave functions on the sphere are spherical harmonics, with effective wave number $k=L/R$. A collective mode is seen clearly; the excitation gap is minimum (≈ 0.075) at $kl \approx 1.4$. The excitation occurs with $L=2,3,\ldots,N$. We identify the large-L (large-k) limit as a well-separated quasiparticle-plus-quasihole pair at mean square (chord) separation given by angular momentum algebra at large N as $2RL/N = kl^2/\nu$. We believe that the small-L (small-k) limit derives from the excitation of a pair of particles from an $m=3$ to an $m=1$ state of relative motion, requiring that $\Delta L \geq 2$. At large k, the excitation energy of the "quasiexciton" must tend to $\Delta^+ + \Delta^-$; Fig. 5 is consistent with this.

Also shown (K) in Fig. 5 is the estimate of $\Delta^+ + \Delta^-$ from low-temperature resistivity measurements by Kawaji *et al.*[8] This estimate is a factor of 3 smaller than our result. Recent numerical calculations by Yoshioka[9] indicate that at most a 20% reduction of the gap can be attributed to mixing with higher Landau levels. We suggest an alternative possible explanation of the discrepancy with the experiments of Ref. 8. A pure $1/r$ potential assumes that $|\psi(z)|^2 = \delta(z)$, where $\psi(z)$ is the wave function describing binding to the surface. Substantial further reduction may be due to spreading of $\psi(z)$ into the surrounding dielectric, which weakens the short-range interaction components. We have found that a significant change of V_1 begins when the width of $\psi(z)$ exceeds $0.1\,l$.

Figure 6 shows the effect of variation of the short-range component V_1, while other pseudopotential

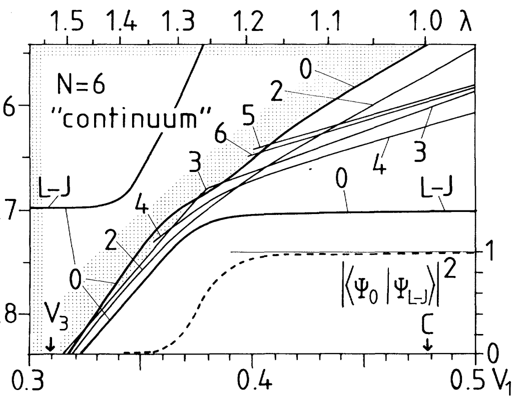

FIG. 6. Low-lying states at $N=6$, $2S=15$ ($\nu=\frac{1}{3}$) as the "hard-core" pseudopotential component V_1 is varied. The other V_m take their Coulomb values. V_3 and the Coulomb value (C) of V_1 are marked. Angular momentum quantum numbers L are indicated. Also shown is the projection of the LJ state on the ground state. In the gapless regime ($\lambda > 1.25$), the LJ state reappears as the *hightest* $L=0$ level.

coefficients are kept at their Coulomb values. A nonvariational explanation of the success of the LJ wave function follows directly from the observation first made in Ref. 2, that it is the *exact ground state* of a truncated "hard-core" pseudopotential V^{hc}, where $V^{\mathrm{hc}}_{m'} = V_{m'} \geqslant 0$ for $m' < m$ and 0 otherwise. Though a formal proof has not yet been found, at $\nu = 1/m$ the gap that separates the LJ ground state of V^{hc} on the sphere from all excited states remains finite as $N \to \infty$. This guarantees the LJ character of the ground state of the potential $V^{\mathrm{hc}} + \lambda(V - V^{\mathrm{hc}})$ at small but finite λ. This stability is equivalent to that of the $\nu=1$ incompressible state against weak interactions that mix in higher Landau levels.

Figure 6 shows that for the Coulomb potential at $\nu=\frac{1}{3}$, the stability limit is $\lambda_c \approx 1.25$, where there is a first-order transition to a gapless state which we find is compressible ($\Delta^+ + \Delta^- \to 0$ as $N \to \infty$). We conjecture this to be the lattice state, though this is difficult to test on the sphere.

While ground-state properties in the incompressible regime are extremely *insensitive* to the lowering of V_1 (the LJ state is the *only* state with this property), excitation energies are *very* sensitive, which suggests that a quantitative comparison to experiment requires realistic modeling of the wave function $\psi(z)$ describing binding of electrons to the interface.

Details of further studies at $\nu=\frac{1}{5}$, $\frac{2}{5}$, and $\frac{2}{7}$ will be given elsewhere. We wish to thank the Aspen center for Physics for its hospitality: The study of the effect of varying the interaction was conceived at the Quantized Hall Effect workshop. One of us (F.D.M.H.) acknowledges receipt of a fellowship from the Alfred P. Sloan Foundation, and he also wishes to thank Dr. C. M. Varma and AT&T Bell Laboratories, where some of the calculations were carried out, for their hospitality. This work was also supported in part by the National Science Foundation under Grant No. DMR-8405347.

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