Spontaneous Emission from an Excited Atom in the Presence of N Atoms and M Modes

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The long-time behavior of the number of photons in a mode of the electromagnetic field is given in closed form in terms of elementary functions when there is initially one (two-level) atom excited in the presence of N unexcited identical atoms and M field modes. The radiation trapping which occurs with only one mode disappears when the number of modes M approaches the total number of atoms in the continuum mode approximation.

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Dicke[1] first emphasized the cooperative nature of the spontaneous emission from a system of identical atoms where the atoms were at equivalent mode positions, e.g., located within a space of linear dimension small compared to a wavelength. The present paper has the purpose of presenting an exact solution for the spontaneous emission of a single atom which is initially excited in the presence of $N-1$ initially unexcited identical atoms, when there are M modes of the electromagnetic field accessible to the radiation. The solution is exact in the sense that the long-time behavior of the average number of photons in a given mode is given in closed form in terms of elementary functions under the conditions that the atoms ($N >> 1$) are at random space positions. Such solutions valid for all times for N-body quantum systems are invariably interesting in their own right because of their expository and pedagogic value. The present model is interesting in that it couples two different quantum systems, N atoms and M field modes. The problem of N two-level atoms with one initially excited atom emitting spontaneously into a single accessible electromagnetic mode has previously been given exactly,[2] when the phenomenon of "radiation trapping"[3,4] is clarified; the atom effectively will not emit its energy as the number of atoms N becomes very large, and the energy is trapped in the single atom. Whether this intriguing phenomenon persists when there are M modes of the field present with frequencies near the atomic resonance will be a question of focus in this paper.

Recently there have been significant advances in cold-cavity techniques with Rydberg atoms[5-7] which bring a number of previously inacessible theoretical predictions (see Haroche[8] for a review) within the purview of the experimentalist. The present paper gives details of spontaneous emission in the presence of $N >> 1$ atoms which is valid for arbitrarily large atom-field coupling and arbitrary times, and it is now reasonable to hope that features of this system will come under experimental scrutiny in the near future. One example of a prediction is the persistence of "ringing" of the total photon number even when the number of modes becomes large, an effect which does not occur when one atom radiates into a large number of modes, when the field buildup is monotonic and closely exponential.

The mathematical setup of the problem has been discussed often before.[1,9,10] The Hamiltonian for the problem is given by ($\hbar=1$)

$$H=\sum_{\mu=1}^{M}\omega_\mu a_\mu^\dagger a_\mu+\Omega\sum_{j=1}^{N}\sigma_j^z+\sum_{\mu,j}\lambda_{\mu j}\sigma_j^{(-)}a_\mu^\dagger+\text{H.c.} \tag{1}$$

The operators satisfy the commutator relations

$$[a_\mu,a_{\mu'}^\dagger]=\delta_{\mu\mu'}, \tag{2}$$

and

$$[\sigma_j^{(+)},\sigma_{j'}^{(-)}]=2\sigma_j^z\delta_{jj'}. \tag{3}$$

The $\lambda_{\mu j}$ are defined as

$$\lambda_{\mu j}=\boldsymbol{\mu}\cdot\mathbf{E}^{(+)}(x_j)/2,$$

where $\boldsymbol{\mu}$ is the electric (magnetic) dipole moment, and $\mathbf{E}^{(+)}$ is the (positive) space part of the electromagnetic field at the position $\mathbf{x}_j$ of the two-level atom. If H is written as H_0+H_1, where $H_1=0$ for $\lambda_{\mu j}=0$, then the matrix elements of H can be expressed in a basis of eigenstates of H_0.

Attention is focused on the situation where the initial energy corresponds to the presence of zero photons in the field, and $N-1$ of the atoms unexcited, so that there is one unit of energy available to be shared among the parts of the system. How is this energy transferred in time? There are $N+M$ states which span the subspace of interest. The base states, eigenstates of H_0, can be defined as follows:

$$|\mu\rangle=|-,\ -,\ldots,\ -;1_\mu\rangle \tag{4a}$$

(one photon is present in mode μ and all atoms are unexcited); and

$$|j\rangle=|-,\ -,\ldots,\ \underset{\underset{j}{\uparrow}}{+},\ldots,\ -;0_\mu\rangle \tag{4b}$$

(one atom, the jth, is excited, all others are unexcited, and there are no photons in the field). The only matrix elements are

$$H_{\mu\mu'} = (\omega_\mu - \Omega)\delta_{\mu\mu'} \equiv 2\Delta_\mu \delta_{\mu\mu'}, \quad (5)$$

and

$$H_{\mu j} = \lambda_{\mu j}, \quad H_{jj'} = 0.$$

(A constant Ω has been subtracted from H with no loss of generality.) The density operator has the form (Schrödinger representation)

$$\rho(t) = \exp(iHt)\rho(0)\exp(-iHt). \quad (6)$$

Interest attaches to the diagonal matrix elements $\rho_{\mu\mu} \equiv n_\mu(t)$, the probability that a photon has been emitted into mode μ at time t, and to ρ_{jj}, the probability that atom j is excited. A calculation of the μj element of H is alone required. This is

$$H^n_{\mu j} = \sum_{\mu'} H_{\mu\mu'} H^{n-1}_{\mu' j} + \sum_{j'} H_{\mu j'} H^{n-1}_{j' j}, \quad (7)$$

and

$$H^{n-1}_{j' j} = \sum_\mu H_{j'\mu} H^{n-2}_{\mu j} = \sum_\mu \lambda^*_{\mu j'} H^{n-2}_{\mu j}. \quad (8)$$

Thus (7) can be written as, with use of Eq. (5),

$$H^n_{\mu j} = 2\Delta_\mu H^{n-1}_{\mu j} + \sum_{\mu', j'} \lambda_{\mu j'} \lambda^*_{\mu' j'} H^{n-2}_{\mu' j}. \quad (9)$$

At this point, use will be made of the reasonable assumption that the atoms are at random space positions $\mathbf{x}_j$, so that

$$\sum_j \lambda_{\mu j} \lambda^*_{\mu' j} = \Lambda^2_\mu \delta_{\mu\mu'}, \quad (10)$$

$$\Lambda^2_\mu = \sum_j |\lambda_{\mu j}|^2. \quad (11)$$

This is most easily seen to be appropriate in the case of free-space modes, when

$$\sum_j^N \lambda_{\mu j} \lambda^*_{\mu' j} = \sum_j \Lambda_\mu \Lambda_{\mu'} \exp[i(\mathbf{k}_\mu - \mathbf{k}_{\mu'}) \cdot \mathbf{x}_j]. \quad (12)$$

With this assumption, Eq. (9) takes the form

$$H^n_{\mu j} = 2\Delta_\mu H^{n-1}_{\mu j} + \Lambda^2_\mu H^{n-2}_{\mu j}. \quad (13)$$

The difference equation (13) has the solution

$$H^n_{\mu j} = \lambda_{\mu j}(h^n_+ - h^n_-)/2\Gamma_\mu, \quad (14)$$

where

$$h_\pm = \Delta_\mu \pm \Gamma_\mu, \quad (15)$$

$$\Gamma_\mu = (\Delta^2_\mu + \Lambda^2_\mu)^{1/2}. \quad (16)$$

Thus the expansion of $\exp(iHt)$ can be summed to give for the probability of finding a photon in mode μ

$$\rho\mu\mu(t) = \sum_{a,b} [\exp(iHt)]_{\mu a} \rho_{ab}(0) [\exp(-iHt)]_{b\mu}$$

$$= |[\exp(iHt)]_{\mu j}|^2, \quad (17)$$

since $\rho_{ab} = \delta_{ab}\delta_{aj}$, where atom j was the initially excited one. Use of Eq. (14) gives

$$\rho_{\mu\mu}(t) = n_\mu(t) = |\lambda_{\mu j}|^2 \sin^2\Gamma_\mu t/\Gamma^2_\mu. \quad (18)$$

For the case that only one mode is accessible to the radiation, Eq. (18) reduces to the expression, in the case that all atoms are in equivalent mode positions,

$$n(t) = \sin^2(N^{1/2}\lambda t)/N, \quad (\Delta_\mu = 0, \quad \lambda_{\mu j} = \lambda). \quad (19)$$

For N very large, no photons are emitted, and the energy remains trapped in the jth atom.[2-4]

The total probability that a photon has been emitted into any mode is the trace over all modes of the elements $\rho_{\mu\mu}$,

$$n(t) = \sum_\mu n_\mu(t) = \sum_\mu |\lambda_{\mu j}|^2 \sin^2(\Gamma_\mu t)/\Gamma^2_\mu. \quad (20)$$

This expression will be examined in the case that $\Lambda^2_\mu = \lambda^2 N$, where N is the effective number of atoms coupled to the field, and λ is independent of μ. Taking the sum over μ to an integral gives

$$n(t) = \int_{-x_0}^{x_0} \lambda\rho(x) \frac{\sin^2[T(x^2+N)^{1/2}]}{x^2+N} dx, \quad (21)$$

where the dimensionless variables $T = \lambda t$, $x = \Delta/\lambda$ are defined, and the number of modes M is

$$\lambda \int \rho \, dx = M. \quad (22)$$

In the case that ρ can be taken as a constant over the range of interest, then the long-time limit of Eq. (21) is easily seen to be

$$n(t) \to (M/2N)[(N^{1/2}/x_0)\tan^{-1}(x_0/N^{1/2})], \quad (23)$$

and is approximately $M/2N$ when x_0 is small compared to $N^{1/2}$.

When x_0 is small compared to $N^{1/2}$, the integral may be well approximated by

$$n(t) \simeq \frac{M}{2N}\{1 - A(t)\cos[\omega_1 t + \phi(t)]\} \quad (24)$$

where $\omega_1 = 2N^{1/2}\lambda t$ and the slowly varying amplitude function A is unity for $t = 0$ and approaches zero as $(N^{1/2}/t)^{1/2}$ when t goes to infinity. It is given in terms of the well known Fresnel integrals C and S by

$$A(z) = \{[C^2(z) + S^2(z)]/z^2\}^{1/2}, \quad (25)$$

where

$$C(z) = \int_0^z \cos(\tfrac{1}{2}\pi s^2)\, ds, \quad (26)$$

$$S(z) = \int_0^z \sin(\tfrac{1}{2}\pi s^2)\, ds, \quad (27)$$

and

$$z^2 = 2\lambda t x_0^2/\pi N^{1/2}. \quad (28)$$

A is shown in the figure as a function of the variable $2\lambda t x_0^2/\pi\sqrt{N}$. The amplitude A falls to zero as $t^{-1/2}$,

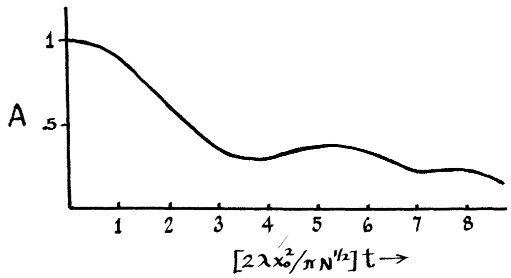

FIG. 1. Amplitude A as a function of time.

since the integrals C and S approach $\frac{1}{2}$ as t approaches infinity.

A calculation similar to the present one and involving only knowledge of $H^n_{\mu j}$ allows calculation of ρ_{jj} which has the asymptotic long-time value

$$\rho_{jj} \to (1-M/N)^2.$$

The conclusion is that "radiation trapping" does not persist as the number of accessible modes approaches the number of atoms, as can be seen from Eq. (23) or (24).

Numerical integration of Eq. (21) in the case that the density of modes is assumed constant over a frequency band $x_0 >> \sqrt{N}$ shows that the oscillatory (or "ringing") behavior seen in the case of $x_0 << \sqrt{N}$ (see Fig. 1) persists. As mentioned before, spontaneous emission into many modes by an isolated atom shows an exponential field buildup, and the ringing is purely a feature of the N-atom nature. Details of this and other features such as the behavior of the atomic system in time for several mode profiles will be given in a forthcoming publication.

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