Is Perturbation Theory the Asymptotic Expansion in Lattice Gauge Theories?

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It is shown that in a gauge theory on an L^d lattice with a compact Lie group, the weak-coupling expansion of any gauge-invariant Green's function may cease to be an asymptotic representation of the true answer as $L \to \infty$. The disagreement is expected to occur at the two-loop level, in non-Abelian models.

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Consider an $O(N)$ nonlinear σ model on an L^d periodic lattice Λ, with partition function

$$Z=\left[\prod_{x\in\Lambda}\int d^dS_x\right]\exp\left[\frac{(a\mu)^{d-2}}{g}\sum_{x\in\Lambda}\sum_{\nu=1}^{d}\mathbf{S}(x)\cdot\mathbf{S}(x+e_\nu)\right]. \tag{1}$$

Here a is the lattice spacing and μ some arbitrary mass inserted to render the coupling constant dimensionless. Hasenfratz pointed out recently that the weak-coupling computation of the spin-spin correlation function $\langle\mathbf{S}(x)\cdot\mathbf{S}(y)\rangle$ is not asymptotic to the true answer as $g\to 0$.[1] Correctly he traced the difficulty to the existence of zero modes, which cannot be treated perturbatively, since in those directions there is no Gaussian damping. Introducing collective coordinates to treat the zero modes, Hasenfratz verified that on a finite lattice the weak-coupling expansion of $\langle\mathbf{S}(x)\cdot\mathbf{S}(y)\rangle$ is asymptotic to the true answer as $g\to 0$. (A similar procedure, and precisely for the same reason, has been used for many years in semiclassical approximations.[2])

Richard and I pointed out that although such a procedure works for finite L, for $d\leqslant 2$ as $L\to\infty$, the perturbative answer is not asymptotic to the true answer for any nonzero g.[3] The reason is the Mermin-Wagner theorem[4]: For $d\leqslant 2$, for any nonzero g, having fixed the spin at some site x_0 to be $\mathbf{S}_0$ does not imply that the spins far away are close to the direction $\mathbf{S}_0$. In fact they can point in any direction with equal probability, and hence the assumption of small fluctuations for small g is incorrect. (That the weak-coupling perturbative answer, while infrared finite,[5] is not asymptotic to the true answer, can be verified in one dimension, where the exact answer can be computed.[6]) An interesting question, to which apparently there is yet no rigorous answer, is what happens for $d>2$, where, as $g\to 0$, one has long-range order. One would guess that weak-coupling perturbation theory is all right. Yet the theorem by McBryan and Spencer[7] about the existence of exponentially decaying correlations in the free, cutoff field model

$$S=\left[\sum_{x\in\Lambda}\sum_{\nu=1}^{d}\right][\phi(x+e_\nu)-\phi(x)]^2,\quad -1\leqslant\phi(x)\leqslant 1,$$

in *any* dimension, makes me rather cautious.

In this Letter I would like to point out that similar difficulties with the weak-coupling perturbation expansion are encountered in lattice gauge theories for continuous groups in any dimension. I begin my discussion on a lattice with L^d sites and free boundaries, the partition function being

$$Z=\left[\prod_{\substack{\nu=1\\x\in\Lambda}}^{d}\int dg_{x,\nu}\right]\exp\left[-\frac{(a\mu)^{d-4}}{2g_0^2}\sum_{x\in\Lambda}\sum_{\substack{\mu,\nu=1\\\mu\neq\nu}}^{d}A_{x,\mu\nu}(g)\right]. \tag{2}$$

Here g is an element of some compact Lie group G and $A_{x,\mu\nu}$ is the gauge-invariant action of the plaquette $x,\mu\nu$:

$$A_{x,\mu\nu}=\tfrac{1}{2}[\chi(g_{x,\mu\nu})+\bar\chi(g_{x,\mu\nu})],\qquad g_{x,\mu\nu}\equiv g_{x,\mu}\circ g_{x+e_\mu,\nu}\circ g_{x+e_\nu,\mu}^{-1}\circ g_{x,\nu}^{-1},$$

where χ is a character on G. One can think of the elements $g_{x,\nu}$ of the group G as spin variables $\mathbf{S}_{x,\nu}$ taking values on some compact manifold $M(G)$ [for example, for O(2) on S^1 and for SU(2) on S^3]. One can then rewrite Eq. (2) as

$$Z=\left[\prod_{\substack{\nu=1\\x\in\Lambda}}^{d}\int d^dS_{x,\nu}\right]\exp\left[-\frac{(a\mu)^{d-4}}{2g_0^2}\sum_{x\in\Lambda}\sum_{\substack{\mu,\nu=1\\\mu\neq\nu}}^{d}A_{x,\mu\nu}(\mathbf{S})\right]. \tag{3}$$

Thus the lattice gauge model is a spin model, with the spins placed on the links and the plaquette action a certain function of the four spins attached to the plaquette, having the property that it is invariant when the group G acts on any two adjacent spins. For example, for O(2) a possible plaquette action is

$$A_{x,\mu\nu} = -\mathbf{S}_{x,\mu}\cdot\mathbf{S}_{x+e_\nu,\mu}\mathbf{S}_{x,\nu}\cdot\mathbf{S}_{x+e_\mu,\nu} - (\mathbf{S}_{x,\mu}\times\mathbf{S}_{x+e_\nu,\mu})\cdot(\mathbf{S}_{x,\nu}\times\mathbf{S}_{x+e_\mu,\nu}). \qquad (4)$$

The local gauge invariance present in the problem means that to compute the expectation value of any gauge-invariant Green's function one can freeze any L^d-1 spins in any position that one chooses (provided that they do not lie on a closed path). For example, let me assume that along the maximal axial tree all spins are frozen in a common direction $\mathbf{S}_0$. Having "fixed the gauge" one is ready to discuss whether perturbation theory is applicable or not. Basically the question is the following: As $g_0 \rightarrow 0$, do the remaining spins point in a small region around $\mathbf{S}_0$? On this finite L^d lattice with free boundaries, the answer is easily seen to be yes. Indeed, as $g_0 \rightarrow 0$, spin configurations in which there is no local alignment such that for each plaquette $A_{x,\mu\nu} \sim O(g_0^2)$ are exponentially suppressed in Eq. (3). [On a finite lattice the entropy ($\leq L^d$) can never compensate the factor $\exp(-c/g_0^2)$ produced by a "frustrated" plaquette as $g_0 \rightarrow 0$, L fixed.] On the basis of this observation, one can immediately verify that on a finite lattice with free boundaries all the spins are as close to $\mathbf{S}_0$ as desired as $g_0 \rightarrow 0$ (see Fig. 1).[8]

The above argument cannot be made for $g_0 << 1$ fixed, $L \rightarrow \infty$ (the entropy could win). In fact, as I shall presently argue, as $L \rightarrow \infty$, for any nonzero g_0, there exist spin configurations such that at a nonzero fraction of points on the lattice Λ, $\langle \mathbf{S}_{x,\mu}\rangle = 0$, and which *a priori* contribute to the partition function at least as much as the region in which $\langle \mathbf{S}_{x,\mu}\rangle = \mathbf{S}_0$, $\forall x \in \Lambda$. To see this consider the spin orientations shown in Fig. 2 for a three-dimensional lattice. By choosing the spins along the broken lines at $\mathbf{S}_0$ (gauge freedom) or around $\mathbf{S}_0$, one has decomposed the original three-dimensional lattice into an infinite number of uncoupled two-dimensional continuous spin models *without "frustrating" the system.* Invoking the Mermin-Wagner theorem for these two-dimensional models proves the assertion (even though along some lines the spins point along $\mathbf{S}_0$, by the choice of the gauge). The generalization of this construction to higher dimensions is immediate.

I make the following remarks:

(i) Several years ago Elitzur[9] proved that in any theory with local gauge invariance $\langle \mathbf{S}_{x,\mu}\rangle = 0$. I am arguing for a stronger statement: If the gauge group is continuous, for any d and any nonzero g_0, $\langle \mathbf{S}_{x,\mu}\rangle = 0$ on most links of an infinite lattice, even after the gauge has been completely fixed to be maximally axial.

(ii) Otherwise said, in a theory with local gauge invariance with a continuous group, there is no long-range order for any d and any $g_0 \neq 0$, even after the gauge has been completely fixed (in the axial gauge).

(iii) Consequently, weak-coupling perturbation theory is unjustified. In particular in two dimensions one can explicitly verify that the weak-coupling perturbative answers are not asymptotic to the true answers, if the gauge group is non-Abelian. [In the axial gauge, the gauge theory model reduces to a one-dimensional spin model with ordinary nearest-neighbor couplings; see, for instance, Eq. (4).]

(iv) Remarkably, both the Green's functions computed in the weak-coupling perturbation theory and the ones obtained by expansion of the exact answers in powers of g_0 have all the desired symmetries. The difference presumably arises because of the different Hilbert spaces employed in the two computations (square integrable on S^2, respectively, R^2).

(v) The proof of the absence of long-range order was given in the axial gauge. (This gauge has been

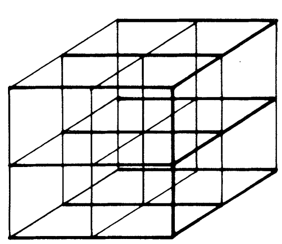

FIG. 1. Typical spin configuration on a lattice with free boundaries. On the links drawn in heavy lines the spins are fixed in the direction $\mathbf{S}_0$. The remaining spins are free. As $g_0 \rightarrow 0$ they will approach $\mathbf{S}_0$ to avoid creating "frustrated" plaquettes.

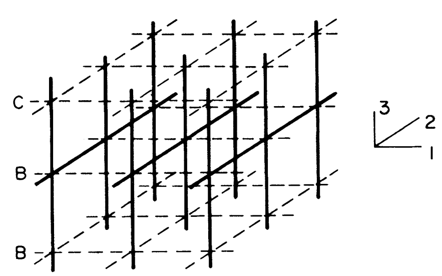

FIG. 2. A spin configuration dominating the trivial one $\mathbf{S}_{x,\mu} \approx \mathbf{S}_0$, $\forall x \in \Lambda$. On the links drawn in broken lines the spins point at $\mathbf{S}_0$ or are allowed to vary around $\mathbf{S}_0$. The remaining spins are allowed to take any value on $M(G)$.

used previously to prove physical positivity for $g_0 \to \infty$,[10] and to do weak-coupling perturbation theory.[11]) It would be desirable to verify my conclusions in some other gauge, perhaps with a numerical investigation. I predict that for non-Abelian groups the expectation value of any gauge-invariant observable measured with the Monte Carlo technique (no gauge fixing) will differ from its (lattice) perturbative value in any gauge, at the two-loop level, as $L \to \infty$.[12]

Could one find a renormalization scheme [let $g_0 = g_0(L) \to 0$ as $L \to \infty$] in such a way that perturbation theory becomes asymptotic to the true answer? Richard and I analyzed this question for continuous spin models and reached a negative conclusion.[3] I believe that the same arguments apply to the present case.

The arguments advanced in this paper *do not* imply that (a) any lattice gauge model with $d > 2$ behaves essentially like its $d = 2$ continuous spin version; indeed spin configurations other than the ones in Fig. 2 may dominate the partition function. Nor do they imply that (b) one cannot do perturbation theory in ordinary (noncompact) QED; indeed in the absence of matter fields (the case discussed in this paper), that theory is purely Gaussian by construction.

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