Puzzling Aspect of Quantum Field Theory in Curved Space-Time

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A ϕ^3 field theory in a six-dimensional conformally flat space-time is studied at the two-loop level. It is found that the state-dependent divergences do not cancel and thus the theory does not become renormalized in the usual way.

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Currently, it is believed by some workers in the field that any theory which is renormalizable in flat space-time is renormalizable in curved space-time. In this Letter, we present the results of an explicit calculation which leads us to believe that this is not a foregone conclusion. A longer account of this work will appear shortly.

We study a ϕ^3 theory in a six-dimensional conformally flat space-time. The metric is

$$ds^2 = \Omega^2(\eta)\left[d\eta^2 - \sum_{i=1}^{5}(dx^i)^2\right], \tag{1}$$

and the Lagrangian density is

$$\mathscr{L} = (-g)^{1/2}[\tfrac{1}{2}g^{\mu\nu}\partial_\nu\phi\,\partial_\mu\phi - \tfrac{1}{2}(m_R^2 + \xi_R R)\phi^2 - \tfrac{1}{2}(\delta m^2 + \delta\xi\, R)\phi^2 - (1/3!)g_B\phi^3], \tag{2}$$

where ϕ is the renormalized field and is related to the bare field ϕ_0 by $\sqrt{Z}\,\phi = \phi_0$. The wave-function renormalization constant and the counterterms are defined by

$$Z(g,n) = 1 + \sum_{\nu=1}^{\infty}\frac{c_\nu(g)}{(n-6)^\nu} = 1 + \sum_{\nu=1}^{\infty}\sum_{r=\nu}^{\infty}\frac{c_{\nu r}g^4}{(n-6)^\nu}, \tag{3}$$

$$m_B^2 Z = m_R^2 + \sum_{\nu=1}^{\infty}\frac{m^2 b_\nu(g)}{(n-6)^\nu} = m_R^2 + m_R^2\sum_{\nu=1}^{\infty}\sum_{r=\nu}^{\infty}\frac{b_{\nu r}g^r}{(n-6)^\nu}, \tag{4}$$

$$\xi_B Z = \xi_R + \sum_{\nu=1}^{\infty}\frac{d_\nu(\xi,g)}{(n-6)^\nu} = \xi_R + \sum_{\nu=1}^{\infty}\sum_{r=\nu}^{\infty}\frac{d_{\nu r}(\xi)g^r}{(n-6)^\nu}, \tag{5}$$

and

$$g_B Z^{3/2} = \mu^{3-n/2}\left[g_R + g_R\sum_{\nu=1}^{\infty}\frac{a_\nu(g)}{(n-6)^\nu}\right] = \mu^{3-n/2}\left[g_R + g_R\sum_{\nu=1}^{\infty}\sum_{r=\nu}^{\infty}\frac{a_{\nu r}g^r}{(n-6)^\nu}\right]. \tag{6}$$

As usual, $m_B^2 = m_R^2 + \delta m^2$ and $\xi_B = \xi_R + \delta\xi$.

The expansion for ξ_B given in Eq. (5) is the general 't Hooft–type renormalization prescription for the renormalization of a parameter that appears in the Lagrangian. In this renormalization scheme ξ and g are both treated as running coupling constants. The other popular renormalization scheme is the one advocated by Collins.[1] Collins picks $\xi_B = \xi(n) \equiv (n-2)/4(n-1)$. The difference between these two ways of treating ξ is just the usual ambiguity in any renormalization formalism. That is, one can add to Eq. (5) terms of order $n-6$ which vanish at the physical dimension. If one takes $\xi_R = \frac{1}{5}$ and adds $\xi(n) - \frac{1}{5}$ to the right-hand side of Eq. (5), one returns to Collin's renormalization scheme. These two renormalization schemes provide two distinct, independent ways of treating ξ.[2] Both renormalization schemes break conformal invariance but the magnitude of the symmetry breaking is different for the two schemes and only experiment can decide which is the correct scheme.[2]

Drummond[3] has renormalized ϕ^3 in six dimensions in a seven-dimensional spherical space-time; however, Drummond took $\xi_B = \xi(n)$ and thus his results *cannot* be compared with ours.

Macfarlane and Woo[4] have renormalized ϕ^3 in six dimensions in flat space-time at the one- and two-loop levels. Here we report the results of our two-loop calculation in a space-time whose metric is given by Eq. (1). The one-loop calculation has been done by Gass[5] and Toms[6] and we use the notation of Gass.[5] We treat both the $m^2\phi^2$ and the $R\phi^2$ terms as operator insertions.

The Feynman rules for the theory have been given by Birrell[7,8] and by Gass[5,9] and we will just state them here. The equation that the Feynman propagator G_F satisfies is

$$[\Box_x + \xi R(x) + (m^2 - i\epsilon)]G_F(x,x') = -\delta^n(x,x')[-g(x)]^{-1/2}. \quad (7)$$

Instead of working directly with G_F, it is easier to define g_F as

$$g_F(x,x') = \Omega(\eta)^{(n-2)/2} G_F(x,x')\,\Omega(\eta')^{(n-2)/2} \quad (8)$$

and work with g_F. The equation g_F satisfies is

$$[\Box_x^2 + m_-^2 - i\epsilon]g_F(x,x') = -\delta^n(x-x') + V(\eta)g_F(x,x'), \quad (9)$$

where

$$V(\eta) \equiv m_-^2 - m^2\Omega^3(\eta) - \left(\xi - \frac{n-2}{4(n-1)}\right)\Omega^3(\eta)R(\eta), \quad (10)$$

$$m_- \equiv m\Omega^{3/2}(\eta = -\infty), \quad (11)$$

and $\Box^2$ is the flat space-time d'Alembertian.

The propagator g_F is the propagator for $\Phi \equiv \Omega^{(n-2)/2}\phi$ and the Feynman rules that we give are for the Φ field. This means that we have an additional (and time-dependent) interaction term because

$$(-g)^{1/2} g_B \phi^3 \to \Omega(\eta)^{3-n/2}(-g)^{1/2} g_B \Phi^3.$$

Therefore, the Φ^3 vertex is not just $g_B\Phi^3$ but is $g_B[1+\frac{1}{2}(n-6)\ln\Omega(\eta)]\Phi^3$. Since the $\ln\Omega(\eta)$ pieces are of order $n-6$, two-loop diagrams with one $\ln\Omega(\eta)$ insertion will have single pole terms containing $\ln\Omega(\mu)$ pieces. It can, however, be shown[10] by explicit calculation that all the $\ln\Omega(\mu)$ divergences cancel among themselves. The Feynman rules are the following: (1) for each vertex, a factor of $-ig[1+\frac{1}{2}(n-6)\ln\Omega(\eta)]$; (2) for each propagator with momentum q, a factor of i/q^2; (3) for each mass insertion, a factor of im_-^2; and (4) for each V insertion into a line carrying momentum q, insert $i\hat{V}(q_0 - q_0')$, multiply by $\exp[i(q_0-q_0')]$, and integrate over q_0', where

$$\hat{V}(p,q) = (1/2\pi)\int_{-\infty}^{\infty} e^{i(p-q)\eta}V(\eta)\,d\eta. \quad (12)$$

Note that our operator insertions involve m_-^2 and V instead of m^2 and R; this is just a matter of convenience. It should also be noted that the V insertion (which will be denoted by a triangle) changes the zero component of the momentum of the line that it is inserted into. This is because the space-time is not invariant under time translations.

Insertions are to be made on the diagram in all possible ways such that the resulting diagram remains divergent. Simple power counting shows that each self-energy diagram will have at most one V or m_-^2 insertion and that there will be no insertions on the vertex diagrams. Hence, it suffices to consider only the self-energy diagrams.

The self-energy diagrams with no insertions are the same as in flat space-time. The divergences in these diagrams can be removed by choosing

$$C_{24} = 5/36(4\pi)^6, \quad (13)$$

and

$$C_{14} = 13/432(4\pi)^6. \quad (14)$$

Those coefficients agree with Macfarlane and Woo's[4] result and with Kounnas's[11] result.

The diagrams with V insertions are shown in Fig. 1. It is obvious that the diagrams with mass insertions will be the same except for the replacement of the V insertions by mass insertions. The only change from the flat space-time case for the mass insertions is that m^2 is replaced by m_-^2.

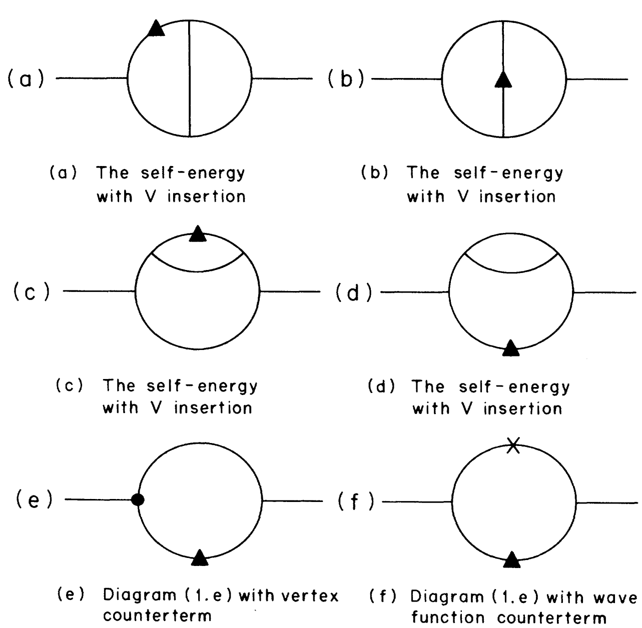

FIG. 1. V insertions on the self-energy at the two-loop level.

The contribution from Fig. 1(a) is

$$S_{41}(p)=\frac{-ig^4}{(4\pi)^6}\left(\frac{p^2}{\mu^2}\right)^{n-6}\left[\frac{V(\eta)}{(n-6)^2}+\frac{(\gamma-3/4)V(\eta)}{(n-6)}+\frac{2\hat{R}(p)}{(n-6)}\right], \tag{15}$$

where

$$\hat{R}(p)=\int dl_0\, e^{il_0\eta}R_1(p,l), \tag{16}$$

and

$$R_1(p,l)=\int_0^1 dx\, dt\, t\ln\{(1-t)+x(1-xt)l^2/p^2-2x(1-t)l\cdot p/p^2\}. \tag{17}$$

The $\hat{R}(p)$ term is a state-dependent divergence. All such divergences must cancel when the diagrams are summed if the theory is to be renormalizable.

The contribution from Fig. 1(b) is

$$S_{44}(p)=\frac{ig^4}{4(4\pi)^6}\left(\frac{p^2}{\mu^2}\right)^{n-6}V(\eta)\frac{1}{n-6}. \tag{18}$$

The contribution from Fig. 1(c) is

$$S_{43}(p)=\frac{ig^4}{2(4\pi)^6}\left(\frac{p^2}{\mu^2}\right)^{n-6}\left(\frac{-1}{(n-6)^2}+\frac{9/4-\gamma}{(n-6)}\right)V(\eta). \tag{19}$$

There is no state-dependent divergence from this diagram. This will spoil the renormalizability of the theory since there will be no state-dependent divergence to cancel against the state-dependent divergence from the conformal coupling-constant renormalization.

The contribution from Fig. 1(d) is

$$S_{42}(p)=\frac{-ig^4}{8(4\pi)^6}\left(\frac{p^2}{\mu^2}\right)^{n-6}\left[\frac{-2V(\eta)}{(n-6)^2}+\frac{(13/6-2\gamma)}{(n-6)}V(\eta)-\frac{4\hat{R}(p)}{(n-6)}\right]. \tag{20}$$

The contribution from Fig. 1(e) is

$$S_{45}(p)=\frac{ig^4}{(4\pi)^6}\left[\frac{2V(\eta)}{(n-6)^2}+\frac{V(\eta)}{(n-6)}\ln\left(\frac{p^2}{\mu^2}\right)+\frac{(\gamma-1/2)}{(n-6)}+\frac{2\hat{R}(p)}{(n-6)}\right]. \tag{21}$$

The contribution from Fig. 1(f) is

$$S_{46}(p)=\frac{ig^4}{2(4\pi)^6}\left[\frac{V(\eta)}{(n-6)^2}+\frac{V(\eta)}{2(n-6)}\ln\left(\frac{p^2}{\mu^2}\right)+\frac{(\gamma-1/2)}{2(n-6)}V(\eta)+\frac{\hat{R}(p)}{(n-6)}\right]. \tag{22}$$

The easiest way to handle the conformal coupling-constant renormalization is to use $Z\xi_B$ instead of ξ_R in the one-loop diagram. When this is done and the mass insertion diagrams are calculated, we find that most of the infinities can be removed by choosing

$$b_{24}=5/4(4\pi)^6, \tag{23}$$

$$b_{14}=7/6(4\pi)^6, \tag{24}$$

$$d_{24}=5(\xi-\tfrac{1}{5})/4(4\pi)^6, \tag{25}$$

$$d_{14}=7(\xi-\tfrac{1}{5})/6(4\pi)^6. \tag{26}$$

However, there still remains a factor of $[ig^4/(4\pi)^6]\times[3V(\eta)/4+\hat{R}(p)]/(n-6)$ and this factor cannot be removed by any of the counterterms in Eq. (2).

The trouble can be traced to the diagram shown in Fig. 1(c). This diagram has no state-dependent divergence and so there is nothing to cancel the state-dependent divergence from the conformal coupling-constant renormalization. The other state-dependent divergences cancel just as one would expect; that is, the pattern of cancellation is the same as for the $\ln(p^2/\mu^2)$ terms.

It should be stressed that the Symanzik identity is satisifed (that is, we have only one Z) as are the 't Hooft pole identities[11]; consequently, our results satisfy all the consistency conditions required of a field theory.

The presence or absence of state-dependent infinities depends on the details of how the momentum flows through the graph. This is why Fig. 1(c) has no state-dependent infinity but Fig. 1(d) does. Presumably this is also why the state-dependent infinities in

ϕ^4 cancel.[7,12,13] We are very surprised at the difference between ϕ^4 in four dimensions and ϕ^3 in six dimensions. *A priori* one would expect them to behave in much the same way, yet ϕ^4 is renormalizable to all orders in any space-time,[13] whereas ϕ^3 is very badly behaved.

It can be argued that this result merely shows that the expansion of ξ_B given in Eq. (5) is incorrect and that one should take $\xi_B = (n-2)/4(n-1)$. With this choice ϕ^3 in six-dimensional conformally flat space-time is renormalizable at the two-loop level. However, Collins[1] has shown that although $\xi = \xi(n)$ works for ϕ^4 when 't Hooft's method[11] is used to sum the divergences, the theory is not finite order by order if $\xi = \xi(n)$. Collins finds that at the four-loop level and beyond, the divergences do not cancel if $\xi = \xi(n)$. Thus it is not clear that $\xi = \xi(n)$ is a significantly better choice.

The implications of our results are unclear. The physically interesting theories in the world are gauge theories and these theories may not be afflicted with the problems ϕ^3 has. For gauge theories one cannot add an R term in any natural way (except, of course, for the Higgs sector). It is true that in nonconformally flat space-times there will be state-dependent divergences even without an R term but the gauge invariance of the theory may force these terms to cancel (see, for example, the calculation by Panaganden[14] and the work by Parker and Toms[15] and the references contained within). We are well aware of the fact that the existence of state-dependent divergences in curved space is at variance with generally held beliefs. If the calculation presented here is indeed correct and complete, the renormalization process in curved space is more subtle and intricate than previously believed. In that case, it would be essential to obtain a deeper, more physical appreciation of the origin of these divergences. It would be especially interesting if gauge invariance would indeed be the mechanism that led to the elimination of these divergences. But such speculations must await the definitive confirmation that the divergences reported here are indeed real and not the result of a calculational error or the omission of essential terms or processes. Since our results satisfy all the known consistency conditions, we do not believe this to be the case.

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